General Relativity
In Four Hours

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Joke on Quantum Gravity

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* In G.R. we can solve the one-body problem analytically, but we can't solve the two-body problem

* In quantum gravity / string theory it isn't even clear that we can solve the zero-body problem!

  We can't even solve for a unique vacuum structure!

Literature

- Geometry & Spacetime : An introduction to General Relativity By Sean Carroll gr-qc/9712019

- General Relativity with Applications to Astrophysics By N. Straumann

- General Relativity : An introduction for Physicists By M. P. Hobson ...

- Gravitation & Cosmology : Principles & Applications of the General Theory of Relativity By Steven Weinberg

Plan & Warning

1| Equivalence Principle

2| Tensors + Reimannian Geometry

3| Geodesics & Newtonian Limit

4| Einstein Field Eq.

5| Application "Cosmological Models

?
\+ Black Holes

Warning: I expect you to know calculus, Special Relativity & Classical Mechanics

Notations "Odd!"

- $a, b, c, d, \ldots, m, n, p, q, \ldots$ will represent space-time in 4-dim (x^0, x^1, x^2, x^3)
- α, β, γ will represent space (x^1, x^2, x^3)
- η_{ab} is the metric of Minkowski space i.e. for S.R. & it given by

$$\eta_{ab} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \qquad c = 1$$

- g_{ab} is metric for curved space with signature $(-,+,+,+)$
- δ_{ab} is the Kroneker delta with

$$\begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$$

I worked horribly strennously, strange that one can endure that ...

Einstein 1915

"Equivalence Principle":

$$\vec{F}^{(1)} = M_i^{(1)} \vec{a}^{(1)} = M_g^{(1)} \vec{g}$$
$$\vec{F}^{(2)} = M_i^{(2)} \vec{a}^{(2)} = M_g^{(2)} \vec{g}$$

$$\vec{a}^{(2)} = \frac{M_g^{(2)}}{M_i^{(2)}} \vec{g}$$
$$\|$$
$$\vec{a}^{(1)} = \frac{M_g^{(1)}}{M_i^{(1)}} \vec{g}$$

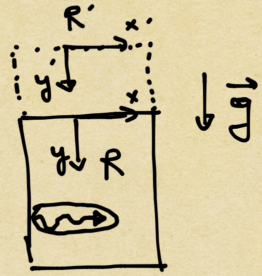
- Checked by Eötvös to an accuracy of about one part in 10^9 (using an experiment based on Earth gravitational field). Recently an experiment based on Sun gravitational field confirmed the equivalence principle to an accuracy of part in 10^{17}.

G.R. 1st Postulate

"In a small Labratory falling in a gravitational field, the Laws of physics are the same as those observed in a Newtonian inirtial frame in the absence of gravitational field."

Implication of the E.P.

* Einstein Elevator: A small idealized elevator, that can perform any kind of vertical motion, including "free fall"



Consider: at $t=0$ R & R' coincide

at $t>0$, elevator accelrates with $\vec{g} = g\vec{j}$

* Elevator is slow, then SR effects can be ignored

* We postulate the "E.P." & we consider that all all inhabitants of elevator have same accelarat°

* Consider an event in the elevator: A light shot from left part of elevator to the right along x-dirc

* Using "E.P." reference R & the particle have same speed ⇒ inertial ⇒ Physics Laws valid

$$\Longrightarrow \boxed{\text{w.r.t. } R : \left.\begin{array}{l} x = ct \\ z = l_0 \end{array}\right\} \text{st. line}}$$

W.r.t. R′ : Since R′ is not inertial for the elevator then we need to use a little trik, Consider an infinitesimal time lapse dt "during which speed of R′ can be assumed constant ⇒ inertial ⇒ Galilian transformation ...

$$\Longrightarrow$$

$$\text{w.r.t. } \boxed{R' : \begin{cases} t' = t \\ x' = x = ct \\ y' = y + g\,dt\,t \end{cases}}$$

Since the elevator is assumed to have const. $\vec{g}$ then the new position is obtained from the old one by means of the same Galilian. trasform

If we sum over small lapse of time $t = \sum dt$.

$\Rightarrow$ w.r.t R' at t

$$\begin{cases} t' = t \\ x' = ct \\ y' = l_0 + \frac{1}{2} g t^2 \end{cases} \Rightarrow y' = \frac{g}{2} x'^2 + l_0$$

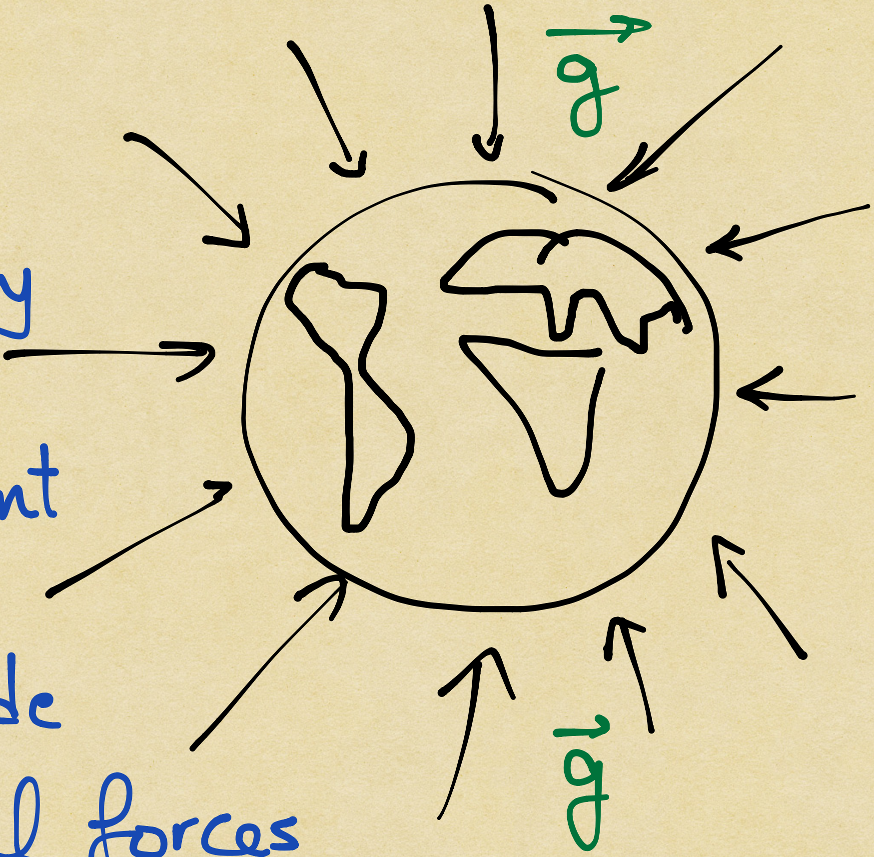
$\Rightarrow$ St. line of the ligt ray become a curved line under, the effect of gravity.

Einstein : Gravity deflects light

$$\boxed{F = ! \frac{GmM}{r^2}}$$

Exercise 1: Repeat the previous prob. for a fast elevator... Leads to "Rindler Space".

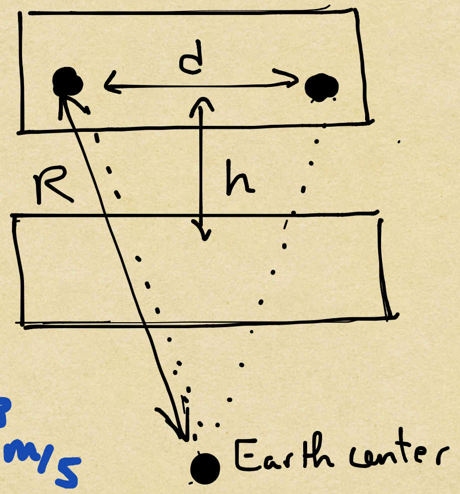
Why the Lab has to be small for the E.P. to work ?



$\vec{g}$ is globally
not const.
its different
in direction
& magnitude
Causing tidal forces
that have tangible effect

Exercise 2 : Free fall huge Lab with 2 particles at d apart & R from Earth. Show that after falling h distance their speed becomes

$$u = \frac{d}{R^2}\sqrt{2GMh}$$



numerical applicat°:

$d = 10\,m$, $h = 10\,Km$, $u = 10^{-3}\,m/s$

Einstein: Is there a coordinate transformat° that can remove the effects of gravity? If moving to an accelarated frame can remove the effect of a uniform gravitational field, Is there a coordinate transformat° that can remove the effects of tidal forces?

A similar mathematical question was answered by B. Reimann in 19th century:

" How to determine weather a certain space is flat or curved ? "

Einstein knew about S.R. & the Geometry of Minkowski Space, but he realised that a more general kind of Geometry must be used that accounts to the curving that Gravity causes. In other words we need to look for mathematics that the theory must have so that E.P. is true

"Geometry, S.R. & More"

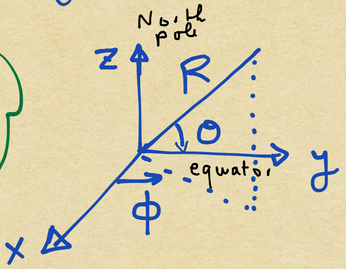
The Geometry of S.R. is the Minkowskian geometry, this geometry has an invariant unit element the "metric"

$$ds^2 = \sum_{mn} \eta_{mn} dx^m dx^n = -c^2 dt^2 + d\vec{x}^{\,2}$$

For Euclidean space: $ds^2 = (dx^1)^2 + (dx^2)^2 + \cdots$

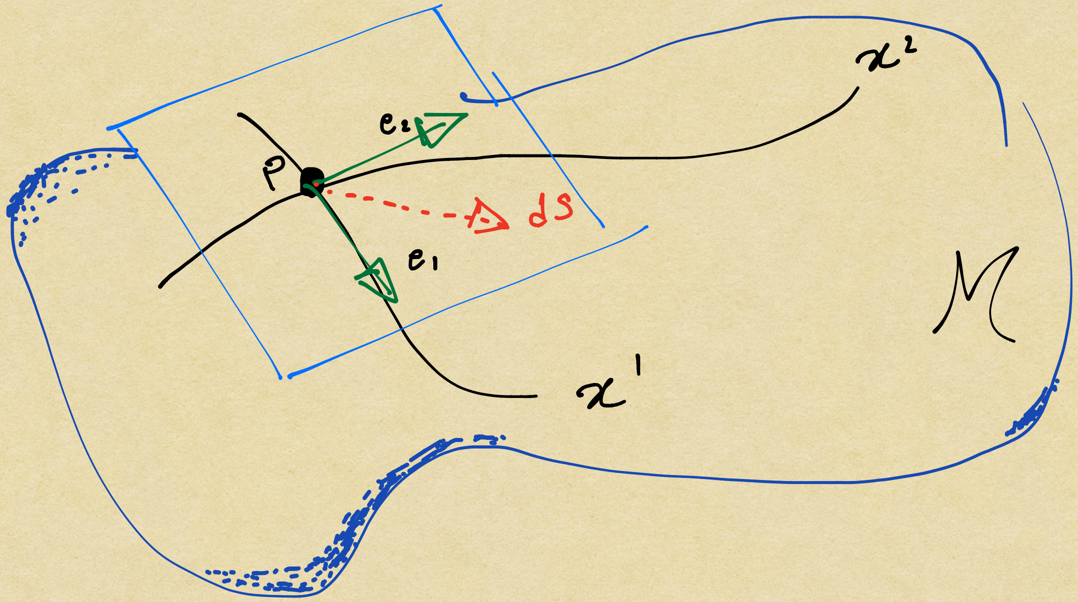
For more general space "e.g. Earth"

$$\Delta S^2 = R^2_{earth}\left(\Delta\theta^2 + \cos^2\theta\, \Delta\phi^2\right)$$



For arbitrary general space, the metric is

$$dS^2 = \sum_{m,n} g_{mn} dx^m dx^n$$



$$\vec{ds} = \sum_a e_a(x)\, dx^a$$

$$ds^2 = \vec{ds} \cdot \vec{ds}$$

$$= \sum_a dx^a e_a \sum_b dx^b e_b$$

$$= \sum_{ab} (e_a \cdot e_b)\, dx^a dx^b$$

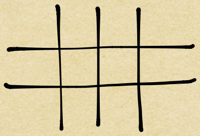
Call $e_a(x)\, e_b(x) = g_{ab}(x)$

☆ g_{mn} are the elements of the set of metric "tensor", they are symmetric, by construct° ($g_{mn} = g_{nm}$), Can take any real value, the # of indpendent g_{mn} depends on dimension of space by the equation $\frac{d(d+1)}{2}$ "10 for 4-D". In general its position dependent $g_{mn} = g_{mn}(x)$. The values of g_{mn} depends on the coordinate grid we choose ∘∘∘

Example: 2-dim flat space

☆ with cartesian coord: → "flat"

$$ds^2 = dx^2 + dy^2$$

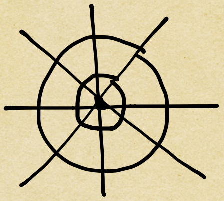


$$[g_{mn}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g_{xx} = g_{yy} = 1 \quad ; \quad g_{xy} = g_{yx} = 0$$

☆ with polar coordinate: "looks curved"

$$ds^2 = dr^2 + r^2 d\theta^2$$



$$[g_{mn}] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$g_{rr} = 1 \quad g_{\theta\theta} = r^2 \qquad g_{r\theta} = g_{\theta r} = 0$$

Curved or flat ??

How to determine weather a space is curved or just appeared curved due to an artifact caused by some unfortunate choice of coordinates?

"Following" Reimann ...

Given :

$$ds^2 = \sum_{mn} g_{mn}\, dx^m dx^n$$

$\exists$? Coordinate transformation that take ds^2 to

$$ds^2 = \sum_{mn} \eta_{mn}\, dx^m dx^n$$

W/t too much guessing, one can notice a vague similarity with Einstein question :

∃ ? A coordinate trasformat° that can remove the effects of non uniform gravity in analogue to the accelarated elevator that removed the effects of uniform gravity.

Later we will see that, this is not just a similarity or analogy, there is a precise correspondance between tidal forces & the curva-ture of space ... to diagnose tidal forces we calc-ulate curvature & for fake gravity curvature is zero

To check if a space is curved or flat we have to learn how ~~~ tranform, which requirs "annoying" mathematics "Tensor Analysis"

"Tensors: Objects that Transform Elegantly"

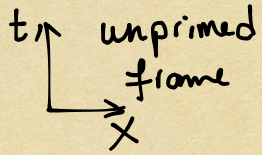
Summation Convention:

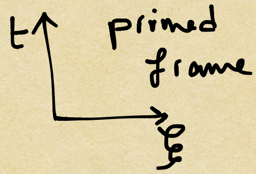
"I have made a great discovery in mathematics
I have suppressed the summation sign every
time that the summation must be made over an
index which occur twice"

Einstein

e.g. : $$ds^2 = \sum_{mn} g_{mn}\, dx^m dx^n = g_{mn}\, dx^m dx^n$$

Consider two Coordinate Grids:





$$x = x(\xi^1, \xi^2, \ldots) = x(\xi)$$
$$\xi = \xi(x^1, x^2, \ldots) = \xi(x)$$

"Scalar: Rank-0 tensor"

e.g. temperature. Value doesn't depend on frame of ref.

$$T(x) = T'(\xi)$$

"Contravariant Vector" (Rank (1,0) tensor)

$$\boxed{(V')^m = \frac{\partial \xi^m}{\partial X^p} V^p}$$

Components of vectors in a certain choice of of coordinate grid

$V = V^m e_m$

Covariant vectors (Rank (0,1) tensor)

$$\boxed{W'_m = \frac{\partial X^p}{\partial \xi^m} W_p}$$

The projection of the vector along the basis:

$V_m = V \cdot e$

Rank (2,0) tensor

let $A^m A^n = T^{mn}$

$$(T')^{mn} = (A')^m (A')^n = \frac{\partial \xi^m}{\partial X^p} A^p \frac{\partial \xi^n}{\partial X^q} A^q$$

$$\Rightarrow \boxed{T'^{mn} = \frac{\partial \xi^m}{\partial X^p} \frac{\partial \xi^n}{\partial X^q} T^{pq}}$$

Rank (2,1) tensor

$$\boxed{T'^{mn}{}_p = \frac{\partial \xi^m}{\partial X^a} \frac{\partial \xi^n}{\partial X^b} \frac{\partial X^c}{\partial \xi^p} T^{ab}{}_c}$$

Useful Tensors Characteristics

1) If all the components of a tensor vanish in one frame of coordinates, they vanish every othe frame of coordinates

2) Once we write an eqn. in a balanced tensorial form, if its true in one coordinate frame, then its true in every other coordinate frame.

Example :

$$R_{ab}(x) - \frac{1}{2} g_{ab}(x) R(x) + g_{ab}(x)\Lambda = \frac{8\pi G}{c^4} T_{ab}(x)$$

$$\Rightarrow R_{ab}(x) - \frac{1}{2} g_{ab}(x) R(x) + g_{ab}(x)\Lambda - \frac{8\pi G}{c^4} T_{ab}(x) = 0$$

$\Rightarrow$ using the first property

$$R'_{ab}(\xi) - \frac{1}{2} g'_{ab}(\xi) R'(\xi) + g'_{ab}(\xi)\Lambda - \frac{8\pi G}{c^4} T'_{\mu\nu}(\xi) = 0$$

$$\Rightarrow R'_{ab}(\xi) - \frac{1}{2} g'_{ab}(\xi) R'(\xi) + g'_{ab}(\xi)\Lambda = \frac{8\pi G}{c^4} T'_{ab}(\xi)$$

g_{mn} is a Tensor:

In relativity time is no longer an absolute parameter, it's just another coordinate ($\vec{r} = \vec{r}(t)$ is non sense in relativity). The absolute parameter is the infiniterimal proper distance (or proper time). In other words ds^2 is the same in all coordinate frames

$$ds^2(x) = ds^2(\xi)$$

$$g_{mn}(x)\, dx^m dx^n = g'_{pq}(\xi)\, d\xi^p d\xi^q$$

$$dx^m = \frac{\partial x^m}{\partial \xi^p} d\xi^p \quad ; \quad dx^n = \frac{\partial x^n}{\partial \xi^q} d\xi^q$$

$$\Longrightarrow$$

$$g_{mn} \frac{\partial x^m}{\partial \xi^p} \frac{\partial x^n}{\partial \xi^q} d\xi^p d\xi^q = g'_{pq}\, d\xi^p d\xi^q$$

(0,2) rank

$$\Longrightarrow \boxed{g'_{pq} = \frac{\partial x^m}{\partial \xi^p} \frac{\partial x^n}{\partial \xi^q} g_{mn}} \Longrightarrow \text{Tensor}$$

Tensor Operations "Things we do on tensor to give new tensor"

1) **Addition:** Only for Tensors with same idices

$$T^{n\cdots}{}_{p\cdots} \mp U^{n\cdots}{}_{p\cdots} = (T \mp U)^{n\cdots}{}_{p\cdots}$$

2) **Multiplication:** Any kind of tensors can be multiplied

I will suppress it

$$V^{mn} \otimes W_p = T^{mn}{}_p$$

← 64 components in 4-D

3) **Raising & Lowering:**

$$\vec{V} \cdot \vec{W} = V^a W_a$$

$$\vec{V} \cdot \vec{W} = V^a e_a W^b e_b$$

$$= V^a W^b g_{ab}$$

Can be done to Non tensors as well

$$g_{ab} W^b = W_a$$

&

$$V^a g_{ab} = V_b$$

4/ Contraction :

$$V^m W_m = T^m{}_m = T \leftarrow \text{number}$$

$$\Rightarrow \quad R^a{}_{bac} = R_{bc}$$

Exercise 3: Show that $R^a{}_{bac}$ is rank $(2,0)$ tensor & $R^a{}_a$ is a scalar.

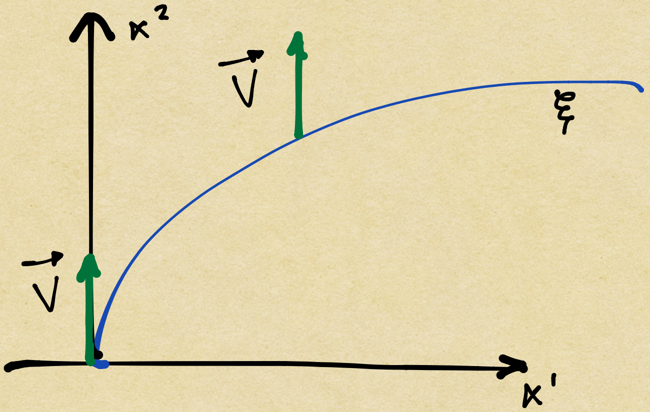
5) Covariant Derivative

Ordinary differentiation of a tensor doesn't give a tensor. $\dfrac{\partial V^m}{\partial x^p}$ is not a tensor

$\partial_p V^m(x) = 0$

$\Rightarrow$ everywhere is zero

However $\partial_r V_m(\xi) \neq 0$



$$\Rightarrow \quad \partial_p V^m \text{ is not a tensor}$$

★ To obtain a tensor out of differentiating a tensor, we need a new def. for the derivative.

★ In arbitrary basis when we compare tensor at two different positions, two things must be taken into consideration, the change in the vector & the change in basis, hence:

$$D_r V^m = \underbrace{\partial_r V^m}_{\text{chang in } V^m} + \underbrace{?_r{}^m}_{\substack{\text{change basis} \\ \Rightarrow \propto \text{ to } V}}$$

$$\Rightarrow$$

$$D_r V^m = \partial_r V^m + \Gamma^m_{rp} V^p$$

★ Γ^m_{rp} is the "Christoffel symbole", often called connection (connects neighbors)

Our Interest: $\Gamma^m_{rp} = \Gamma^m_{pr}$

The Christoffel symbol eq.:

Γ^{c}_{ab} is the term we added to account for change in basis $\Longrightarrow$

$$\boxed{\frac{\partial \vec{e}_a}{\partial x^b} = \Gamma^{c}_{ab}\, \vec{e}_c}$$

Now consider:

$$\partial_c g_{ab} = \partial_c(\vec{e}_a \cdot e_b)$$

$$= \partial_c e_a \cdot e_b + e_a \cdot \partial_c e_b$$

$$= \Gamma^{d}_{ac}\, e_d\, e_b + e_a\, \Gamma^{d}_{cb}\, e_d$$

$$= \Gamma^{d}_{ac}\, g_{db} + \Gamma^{d}_{bc}\, g_{ad} \quad \ldots (1)$$

Cyclically permute that

$$\partial_a g_{bc} = \Gamma^{d}_{ba}\, g_{dc} + \Gamma^{d}_{ca}\, g_{bd} \quad \ldots (2)$$

$$\partial_b g_{ca} = \Gamma^{d}_{cb}\, g_{da} + \Gamma^{d}_{ab}\, g_{cd} \quad \ldots (3)$$

$\Longrightarrow$ (1) + (2) − (3) gives

$$2\Gamma^{d}_{cb}\, g_{ad} = \partial_c g_{ab} + \partial_b g_{ca} - \partial_a g_{bc}$$

using $\boxed{g^{ea} g_{ad} = \delta^{e}{}_{d}}$ & relabelling

The Connection becomes finally :

$$\Gamma^{a}{}_{bc} = \frac{g^{ad}}{2} \left(\partial_b g_{ac} + \partial_c g_{bd} - \partial_d g_{bc} \right)$$

* Exercise 4 : Show that $e^a . e_b = \delta^a{}_b$
hence deduce that $g^{ac} g_{cb} = \delta^a{}_b$

* Exercise 5 : Why the connection is not a tensor ?... Can you show it using coordinate transformation ?

* Exercise 6 : Calculate the connection elements for the metric

$$ds^2 = dr^2 + r^2 d\theta^2$$

Can you make any conclusion abt the space ?

Exercise 7: Show using

$$D_m T^{rp} = \partial_m T^{rp} + \Gamma^{mr}_q T^{qp} + \Gamma^{mp}_q T^{rq}$$

that the metric tensor is covariantly constant

$$D_r g^{mn} = 0$$

Exercise 8: Show that the covariant derivative of covariant vector is:

$$\boxed{D_r V_m = \partial_r V_m - \Gamma^p_{rm} V_p}$$

Deduce that the Covariant derivative of 2-tensor is given by:

$$D_m T_{rp} = \partial_m T_{rp} - \Gamma^q_{mr} T_{qp} - \Gamma^q_{mp} T_{rq}$$

"At present I occupy myself exclusively with the problem of gravitation & now believe that I s all master all difficulties with the help of a friendly mathematician here Maral Grossmann). But one thing is certain, in all my life I have never laboured nearly as hard, & I have become imbued with great respect for mathematics. The subtler part of which I had in my simple-mindedness regarded as pure luxury until now. Compared with this problem, the original relativity is child's play."

(Einstein in a letter to A. Sommerfeld)

CURVATURE

☆ Our main conclusion of the introduct° was that the question of finding out weather there is a real gravitational field or not is identical to the problem of finding if a certain smooth metric geometry is flat or not...

☆ Given an arbitrary space, to know weather the space is flat or curved, we can do it either by:

1: Long Bad Way

Look over all change of coordinates & find if there one that $g_{mn} \longrightarrow \eta_{mn}$

or 2. Look for a diagnostic quantity that characterize the space, independent of the basis chosen & Calculate it. The value will tell us if the space is flat or Curved.

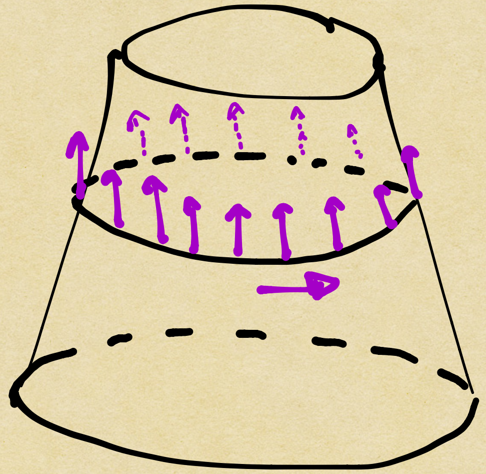
The quantity we are after is the Curvature tensor "$R^d{}_{abc}$". Also called Reimann tensor. note: should not be confused w/ curve

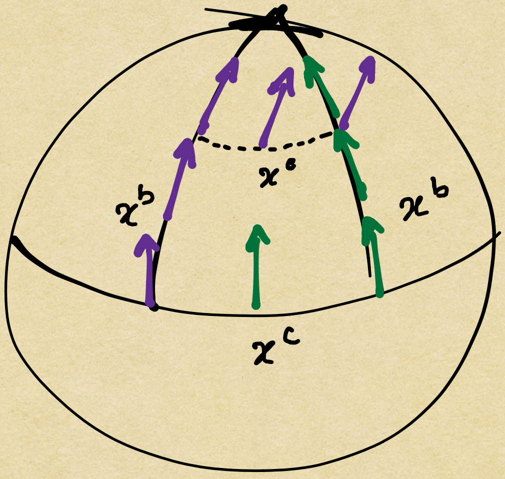
$$\text{if} \quad R^d{}_{abc} = 0$$
$$\Rightarrow \text{Space is flat}$$

Curvature is an intrinsic property of a space. A flat paper can be bent into a cone or cylinder w/t distorsion, but can never be bent into a sphere w/t distorting or cumpling...

A cone & cylinder have zero curvature, However a sphere has a +ve curvature.

Reimann Tensor





Moving a vector along x^r direct° w/t changing it (parallel tramport) is described by $D_s V_m$

By Def :

$$D_c D_b V_a - D_b D_c V_a = R^d{}_{abc} V_d$$

Exercise 6 : Show that the curvature tensor is given by

$$\boxed{R^d{}_{abc} = \partial_b \Gamma^d{}_{ac} - \Gamma^e{}_{ab} \Gamma^d{}_{ec} - (b \leftrightarrow c)}$$

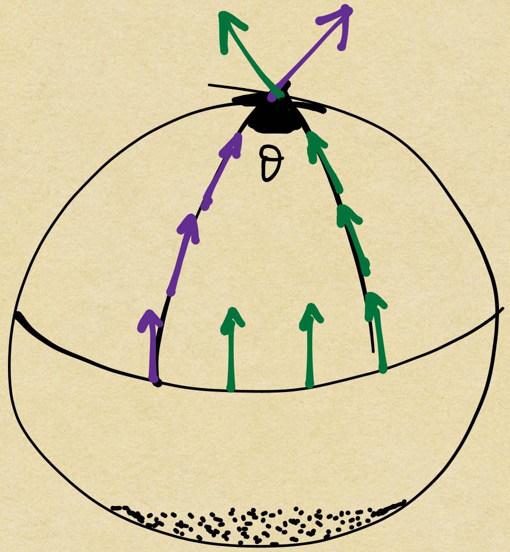
★ Tidal forces stretch & squach, they have bigger effect on bigger objects.

Curvature has exactly same effect as particle moves

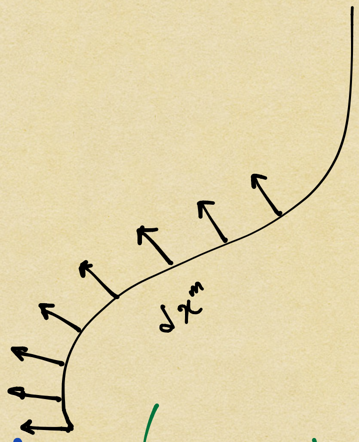
& this effect depends on the size of moving object

Parallel Transport

* Its moving a vector along a path while keeping it constant. "Its path dependent"
in general



* $D_m V^n \, dx^m = DV^n = 0$
The small chang in V^n by going from one pt to its neighbor (covariant differential change in V)

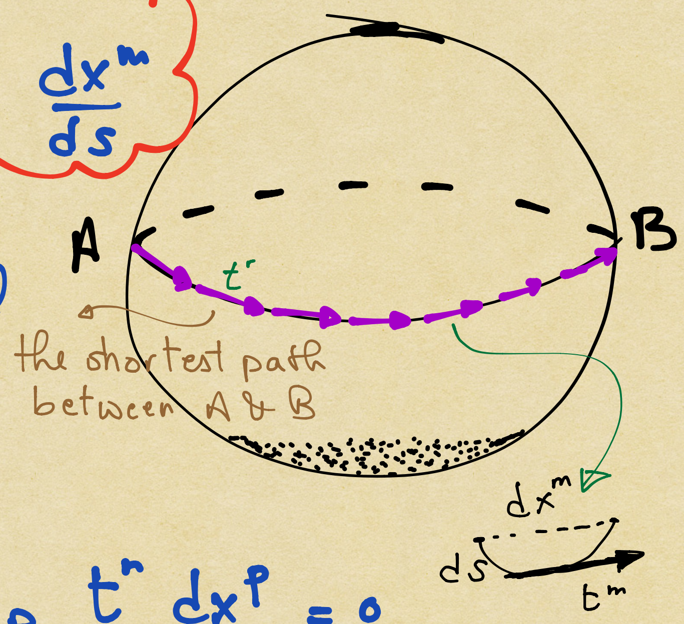


Equation

$$DV^n = dV^n + \Gamma^n_{mr} V^n dx^m = 0$$

Geodesics: Extremest curve between two pts (the path on which the unit tangent is kept parallel to itself)

* unit tangent $\boxed{t^m = \dfrac{dx^m}{ds}}$



* when we parallel transport t^m

$$\Rightarrow Dt^m = 0$$

$$\Rightarrow dt^m + \Gamma^m_{np} t^n dx^p = 0$$

$$\Rightarrow \frac{dt^m}{ds} + \Gamma^m_{np} t^n \frac{dx^p}{ds} = 0$$

Geodesic eq:

$$\frac{d^2x^m}{ds^2} + \Gamma^m_{np} \frac{dx^n}{ds}\frac{dx^p}{ds} = 0$$

G.R. 2nd Postulate:

A free body pursues a time-like geodesic in space-time.

$\Rightarrow$ In the vicinity of gravitating body a particle moves in the straightest way possible $\Rightarrow$ Eq. of motion for a particle is

$$\underbrace{\frac{d^2x^m}{ds^2}}_{\text{accalaration}} = \underbrace{-\Gamma^m_{np}\frac{dx^n}{ds}\frac{dx^p}{ds}}_{\text{force / mass}} \quad \Big\} \text{ replaces Newton's } 2^{nd} \text{ law}$$

Exercise 6: Show that the geodesic eq. gives the Newtonian theory of gravity

$$\vec{a} = -\vec{\nabla}\phi$$

in the limit of slow motion in a weak static field.

Newtonian limit of the Geodesic

To apply Newtonian limit for G.R. eqs.
3. requirements must be satisfied

1. The particles are moving solwly (w.r.t.c)
2. The gravitational field is weak
3. Fields are static (unchanging w/ time)

1^{st} req. $\Rightarrow$ $\dfrac{dx^i}{d\tau} << \dfrac{dt}{d\tau}$

$\Rightarrow$ geodesic boils down to

$$\frac{d^2x^m}{d\tau^2} + \Gamma^m_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

3^{rd} req.

$$\Gamma^m_{00} = \frac{1}{2} g^{mp}\left(\partial_0 g_{p0} + \partial_0 g_{0p} - \partial_p g_{00}\right)$$
$$= -\frac{1}{2} g^{mp}\, \partial_p g_{00}$$

2^{nd} req. (gravitational field is weak)
$\Rightarrow$ can be considered a perturbation of flat space .... Then we can decompose the metric into Minkowski form plus small perturbat°

$$g_{mn} = \eta_{mn} + h_{mn} \qquad |h_{mn}| < 1$$

"g^{mn} is inverse of g_{mn}"

$$\Rightarrow g^{mn} = \eta^{mn} - h^{mn}$$

$\leftarrow$ eq. to 1^{st} order in h

$$h^{mn} = \eta^{mp} \eta^{nq} h_{pq}$$

$$\Rightarrow \frac{d^2 x^m}{d\tau^2} = \frac{1}{2} \eta^{mn} \partial_n h_{00} \left(\frac{dt}{d\tau}\right)^2$$

For $n=0$ $\partial_0 h_{00} = 0 \Rightarrow \frac{d^2 t}{d\tau^2} = 0$

$$\Rightarrow \frac{dt}{d\tau} = \text{const.}$$

For $n = \mu$ (space like component)

$\eta^{\mu\nu}$ = Identity matrix

$$\Rightarrow \quad \frac{d^2x^\nu}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \partial^\nu h_{00}$$

$$\frac{dx^\nu}{d\tau} = \frac{dx^\nu}{dt}\cdot\frac{dt}{d\tau}$$

$$\Rightarrow \frac{d}{d\tau}\left(\frac{dx^\nu}{d\tau}\right) = \frac{d^2x^\nu}{dt^2}\left(\frac{dt}{d\tau}\right)^2 + \underbrace{\frac{dx^\nu}{dt}\cdot\frac{d^2t}{d\tau^2}}_{=0}$$

$$\Rightarrow \qquad \frac{d^2x^\nu}{dt^2} = \frac{1}{2}\partial^\nu h_{00}$$

Comparing with Newton's theory of Gravity $\qquad \dfrac{d^2x^\nu}{dt^2} = -\partial^\nu \phi$

The two eqs. are the same if:

$$h_{00} = -2\phi$$

$\therefore$

$$g_{00} = -(1+2\phi)$$

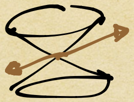
$$g_{00} = -\left(1-\frac{2MG}{r}\right)$$

(Space, Time, Light) Like

My g_{mn} Signature

$$(-\quad +\quad +\quad +)$$

The -ve eigenvalue represents time & the three +ve represents space, hence if:

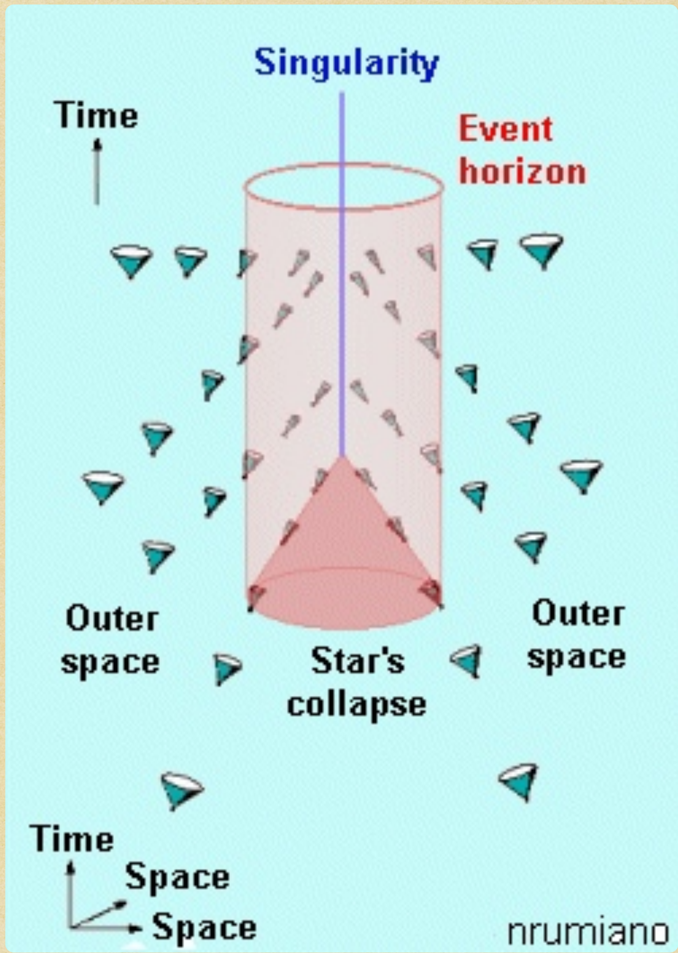
1| $ds^2 > 0 \Rightarrow$ Space like 

2| $ds^2 = 0 \Rightarrow$ light like 

3| $ds^2 < 0 \Rightarrow$ time like 

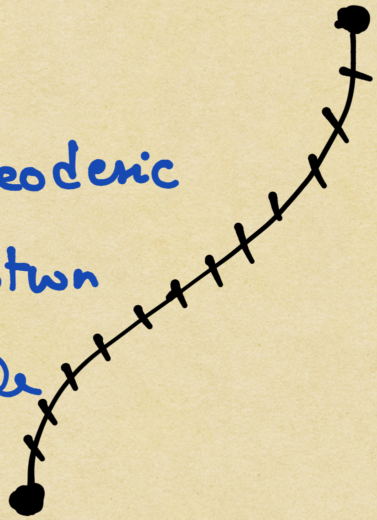
Define $d\tau^2 = -g_{mn}\,dx^m dx^n > 0$

* In space time at every pt we have a metric ∀ every pt. the signature of the metric should be the same (but not necessarly the values) so at every pt. we have a light cone of different shape each depending on the metric entries value at each pt.



Light Cones Shapes near Schwarzschild black hole

The Lagrangian description of a free particle in G.R.



* Another definision of geodesic = stationary distance btwn two pts (where a particle acheives a max or min value) $\Longrightarrow$ Requires minimum time

$\Longrightarrow$ Extrenalize

$$\tau = \int_1^2 \sqrt{-g_{mn}\,dx^m dx^n}$$

* $$\underbrace{\text{Action}}_{\text{energ. time}} = \int \underbrace{\mathcal{L}}_{\text{Energy}}\, \underbrace{dt}_{\text{tim}}$$

$\Longrightarrow$

$$\text{Action} = -mc^2 \int_1^2 \sqrt{-g_{mn}\,dx^m dx^n}$$

$\Longrightarrow$

$$\text{Action} = -mc^2 \int \sqrt{-g_{mn}(x) \frac{dx^m}{dt} \frac{dx^n}{dt}}\, dt$$

The Lagrangian density for a free particle in gravitational field is:

$$\mathcal{L} = -mc^2 \sqrt{-g_{mn}\, \dot{x}^m \dot{x}^n}$$

* Now we can use the Euler Lagrange eq. to determine the eq. of motion, Hence by solely knowing the metric, we can derive the motion of particle.

Exercise 7: Use E.L. eq. to show that the trajectory of the particle is a geodesic "Hint: Use $\mathcal{L}^2$ insted of $\mathcal{L}$"

$$\text{E.L. eq.} : \quad \frac{\partial \mathcal{L}}{\partial X^p} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}^p}$$

$$\mathcal{L}^2 = -g_{mn}\dot{X}^m\dot{X}^n \quad \Longrightarrow \frac{\partial \mathcal{L}^2}{\partial X^p} = -\partial_p g_{mn}\dot{X}^m\dot{X}^n$$

$$\frac{\partial \mathcal{L}^2}{\partial \dot{X}^p} = -g_{mn}\delta^m_p\dot{X}^n - g_{mn}\dot{X}^m\delta^n_p$$

$$= -g_{pn}\dot{X}^n - g_{mp}\dot{X}^m = -2g_{pn}\dot{X}^n$$

$$\frac{d}{dt}\left(-2g_{pn}\dot{X}^n\right) = -2\partial_q g_{pn}\dot{X}^q\dot{X}^n - 2g_{pn}\ddot{X}^n$$

$$= -\partial_q g_{pn}\dot{X}^q\dot{X}^n - \partial_n g_{pq}\dot{X}^n\dot{X}^q$$

$$- 2g_{pn}\ddot{X}^n$$

$$-\partial_p g_{mn}\dot{X}^m\dot{X}^n + \partial_q g_{pn}\dot{X}^q\dot{X}^n + \partial_n g_{pq}\dot{X}^n\dot{X}^q$$

$$+ 2g_{pn}\ddot{X}^n = 0$$

$$\Longrightarrow 2g_{pn}\ddot{X}^n + \left(\partial_q g_{pn} + \partial_n g_{pq} - \partial_p g_{qn}\right)\dot{X}^q\dot{X}^n = 0$$

Einstein Field Equations

Wheeler: Spacetime tells matter how to move, matter tells spacetime how to curve

Warning: Basic Principles are simple, Calculations are pain

For G.R. to be true, we must have a # of limits

$$
\begin{tikzcd}
\boxed{\text{General Relativity}} \arrow[r, "\text{Weak field}", "\text{limit}"'] \arrow[d, "g \to 0"] & \boxed{\text{Newton Gravity}} \arrow[d, "g \to 0"] \\
\boxed{\text{Special Relativity}} \arrow[r, "\text{low}", "\text{Speed}"'] & \boxed{\text{Galilian Physics}}
\end{tikzcd}
$$

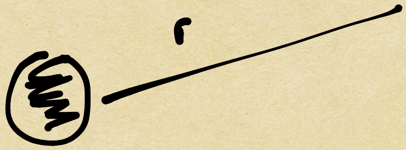
"Newton Gravity is one component of the real gravity"

* Newton didn't think about field, however they can be formalized interms of fields

$$\vec{F} = -m \vec{\nabla} \phi(x) = m \vec{a}$$

$$\Rightarrow \vec{a} = -\vec{\nabla} \phi(x)$$

⇝ Fields tell particles how to move

$M = \int \rho \, dV$ 

* $$\nabla^2 \phi = 4\pi \underbrace{G \rho(x)}$$

outside solution

Matter tells Gravitation field how to look like

$$\phi = -\frac{MG}{r}$$

* Using $g_{00} = 1 + 2\phi$

$$\Rightarrow \nabla^2 \underbrace{g_{00}} = 8\pi G \rho$$

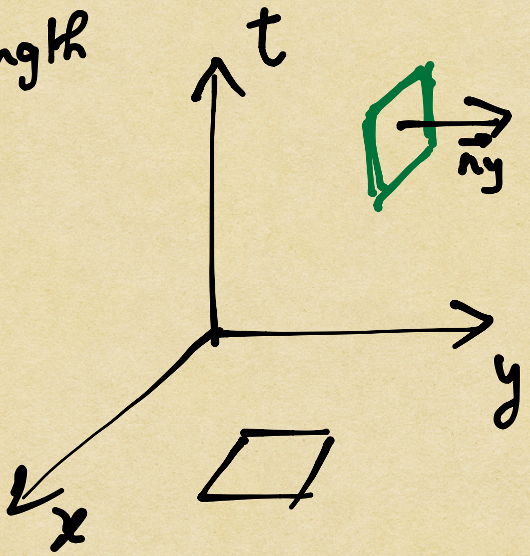
Geometry is determind by matter!

* The last equation which is purely derived from Newton theory implies that matter decide the geometry $\rightarrow$ We need to study Matter i.e. Energy & Momentum & have them in tensorial eq. so that the final form of the eq. be covarian in any frame of ref.

Density, flow & the continuity eq.

$$\sigma = \frac{Q}{\text{Volume}} \rightarrow \text{3 length}$$

"charge per unit volume"



Current or Charge flow

"charge per unit area per unit time"

can be also thought of as a form of charge density

$$j^\mu = \frac{Q}{\text{Area.time}} \rightarrow \text{3 length !}$$

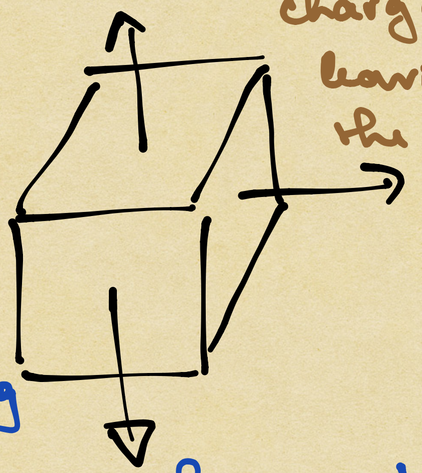
⋆ $\Longrightarrow$ j^μ & σ are very similar creatures

they form 4-vector

$$j^\mu = (\sigma, j^1, j^2, j^3)$$

The "Local" Conservation of Charges

The locality means that in my lab when charges disappeared they are actually leaving through the walls in form of current (that flow through the walls)" so actually its not conservation in a quantified sense".



However, "if we are really curious & we want to know what happened to charge disappeared from my Lab we have to look Globaly "The charge that disappeared in my Lab in Edinburgh has appeared simultaneously, perhaps, in Petnica"

* The local conservation of charges or the continuity eq. is given by

$$-\dot{\sigma} = \vec{\nabla} \cdot \vec{j}$$

$$\Rightarrow \quad \partial_m j^m = 0$$

Energy Momentum Tensor

* Energy can be described interms of density
$\Rightarrow$ in analogy with charge, it can flow & it has continuity eq.

* Also momentum can be described in terms of density, so we have a momentum flow & continuity eq. for momentum

* However, there is an important difference;
Charge by itself is invariant, however Energy by itself is not, also momentum by itself is not invariant, but the 4-vector (E, P^{μ}) is invariant

e^- for all ref. frame is the same

$P_n P^n$ for all ref. frame is the same

$$e^- \to \sigma,\ j \quad \Big\} \text{continuity eq}$$

$$P^n = (E, P_1, P_2, P_3)$$

$E \to \underbrace{T^{00} \quad T^{0\mu}}_{\text{continuity eq.}}$: $D_n T^{0n} = 0$

$P_1 \to \ldots$: $D_n T^{1n} = 0$

$P_2 \to \ldots$: $D_n T^{2n} = 0$

$P_3 \to \underbrace{T^{30} \quad T^{3\mu}}_{\text{continuity eq}}$: $D_n T^{3n} = 0$

$$D_n T^{mn} = 0$$

Indices of T^{mn} :

T^{00} → density part of energy

↓ we are talking abt energy

"one time comp. of smth which is itself is a time comp

$T^{0\mu}$ → flow (current) part of energy

↓ energy

"three space comps of time like object"

$$D_n T^{0n} = 0$$

$T^{\mu 0}$ → density part of momentum

↓ We are talking about momentum

"three zero components of three space like objects"

$T^{\mu\nu}$ → flow part of momentum

↓ momentum

"9 space components of 3 space-like objects"

$$D_n T^{\mu n}$$

$$\Rightarrow \boxed{D_n T^{mn} = 0}$$

* Consider T_{12} & T_{21}

T_{12}: y flow of P_x $\quad$ T_{21}: x flow of P_y

$\Rightarrow$ $T_{21} = T_{12}$

$$\Rightarrow \quad \boxed{T_{mn} = T_{nm}}$$

Energy-momentum tensor is symmetric

Example: Energy-Momentum tensor of the universe on a large scale

The universe is not empty "not sure!". We usually model the matter & energy in the universe by a perfect fluid (by def. a perfect fluid is a continuum of matter that can be descri-bed completely by its pressure (P) & energy density plus it look isotropic in its rest frame).

* In the rest frame, the fluid is at rest & isotropic

Isotropic $\Rightarrow$ T_{ab} is diagonal
& $T_{11} = T_{22} = T_{33}$

at rest $\Rightarrow$ $U^{\mu} = (1,0,0,0)$

* for $ds^2 = -dt^2 + g_{\mu\nu} dx^{\mu} dx^{\nu}$

$$\Rightarrow \quad T_{ab} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{\mu\nu}\, p & \\ 0 & & & \end{pmatrix}$$

$p \rightarrow \dfrac{\text{Momentum}}{\text{Area}.t}$ = Momentum density = pressure

Exercise:

In perfect fluid, all particles have equal velocity in any fixed inertial frame, Show (argue) that in general the Energy-momentum is given by:

$$T_{\mu\nu} = (p + \rho)\, U_{\mu} U_{\nu} + p\, g_{\mu\nu}$$

ρ & p are the energy density and pressure.
U^{μ} is the 4-velocity of the fluid.

* Recall that $\nabla^2 g_{00} = 8\pi G\rho$

But $\rho = T_{00}$ (e.g. $V_x = W_x \Rightarrow \vec{V} = \vec{W}$), Hence $\nabla^2 g_{00} = 8\pi G\rho$ is zero-zero comp. of a tensor eq. that has, $8\pi G\, T_{ab}$, on one side of the equal sign.

$$\Rightarrow \quad ?_{ab} = 8\pi G\, T_{ab}$$

☆ If we denote the L.H.S. $G_{\mu\nu}$ =>

$$\boxed{G_{ab} = 8\pi G\, T_{ab}}$$

G_{ab} describes Geometry

$G_{ab} = G_{ab}(\nabla^2 g_{\mu\nu}, \ldots)$

$G_{ab} = G_{ba}$

$D_a G^{ab} = 0$

What is G^{ab}?

We've seen that the real Reimann geometry of manifold "Space that are locally flat" is characterized by a metric, connection & curvature

$R^a{}_{abc}$ $\qquad g_{ab} \qquad \Gamma^a{}_{bc}$

"Not tensor"

$R^d{}_{abc}$: determine everything about geometry "zero everywhere, flat everywhere"

$\Rightarrow$ G_{ab} must be related to it.

Properties of R_{abcd}

If we sub. in $R_{abcd} = g_{af} R^f{}_{bcd}$ the connection, we get

$$\Rightarrow R_{abcd} = \frac{1}{2}\left(\partial_d \partial_a g_{bc} - \partial_d \partial_b g_{ac} + \partial_c \partial_b g_{ad} - \partial_c \partial_a g_{bd}\right) - g^{ef}\left(\Gamma_{eac}\Gamma_{fbd} - \Gamma_{ead}\Gamma_{fbc}\right)$$

Symmetries in R_{abcd}

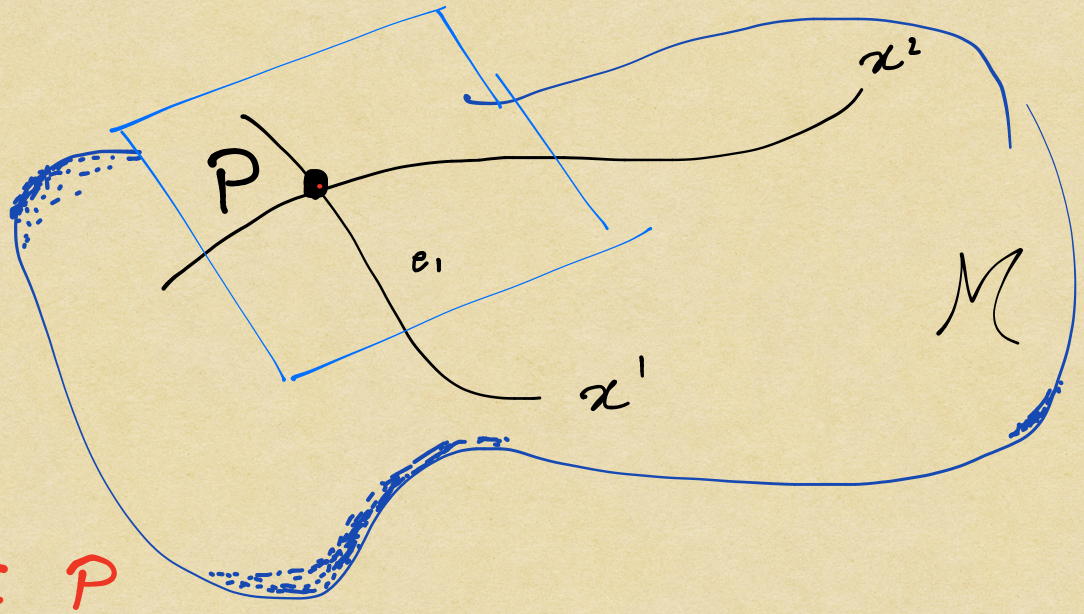
To know the symmetries, enough to examine 1^{st} part

$\therefore$ $$\boxed{\begin{aligned} R_{abcd} &= -R_{bacd} \\ R_{abcd} &= -R_{abdc} \\ R_{abcd} &= R_{cdab} \end{aligned}}$$

Manifold : Topological space that's locally flat

* Using the fact that our manifold is locally flat " Geometrically, means that at each pt. in space we can approximate our space, to linear order, w/ tangent space at each pt.). Then locally at any pt. "say P" means

$$\left\{ \begin{aligned} & g_{mn} = \eta_{mn} \\ & \partial_p g_{mn} = 0 \\ & \partial^2_p g_{mn}, \partial^3 g_{mn}, \ldots \neq 0 \end{aligned} \right.$$



at P

$$R_{abcd}\Big|_P = \frac{1}{2}\left(\partial_d\partial_a g_{bc} - \partial_d\partial_b g_{ac} + \partial_c\partial_b g_{ad} - \partial_c\partial_a g_{bd}\right)_P$$

☆ A property that is valid for R_{abcd} in the in the tangent space ref. at P should be valid any where, then we can use the reduced form of R_{abcd} at P & find the other properties

Exercise 8: Using the reduced eq. for R_{abcd} show R_{abcd} satisfy the "Cyclic Identity"

$$\boxed{R_{abcd} + R_{acdb} + R_{adbc} = 0}$$

& the "Bianchi Identity"

$$\boxed{D_e R_{abcd} + D_c R_{abde} + D_d R_{abec} = 0}$$

Exercise 9: Using the symmetries & properties of R_{abcd}, show that in d-spacetime, the d^4 component of R_{abcd} reduces to $\frac{d^2(d^2-1)}{12}$

Ricci Tensor:

$$R_{ac} = R^b{}_{abc}$$

From Cyclic identity

$$R_{ac} = R_{ca}$$

Exercise 10: Show that

$$R_{ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} - \Gamma^d_{ac}\Gamma^c_{db} + \Gamma^d_{ab}\Gamma^c_{dc}$$

Ricci Scalar

$$R = g^{ab} R_{ab} = R$$

☆ Gathering all the symmetric geometric objects that we have

$$g_{ab}\ ,\ R_{ab}\ ,\ g_{ab} R$$

$$\Longrightarrow \quad G_{ab} = \alpha R_{ab} + \beta g_{ab} R + \Lambda g_{ab}$$

Exercise 12: Show using $D_a G^{ab} = 0$ that $\alpha = 1$, $\beta = -\frac{1}{2}$, Λ arbitrary

$$\Rightarrow \boxed{G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab}}$$

Putting everything together

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 8\pi G\, T_{\mu\nu}$$

No Source Eq.

$$R_{ab} - \frac{1}{2} g_{ab} R = 0$$

$$\Rightarrow R_{ab} = \frac{1}{2} g_{ab} R$$

$$R_a^{\ b} = \frac{1}{2} \delta_a^{\ b} R$$

$$\Rightarrow R_a^{\ a} = \frac{1}{2} \delta_a^{\ a} R$$

$$\Rightarrow R = 2R$$

$$\Rightarrow \boxed{R = 0}$$

$$\Rightarrow \boxed{R_{ab} = 0} \quad (\not\Rightarrow R_{abcd} = 0)$$

The space is called Ricci flat, doesn't mean that the space is flat (the curvature tensor is not zero).

Solutions for this eq. is, for example, the Schwarzchild black hole, which is the metric for a space outside a spherical massive body (nthg in the space other than the spherically symmetric B.H.)

Also gravitational waves is another example of solutions for Ricci flat spaces.

* You will be convinced of the general theory of relativity once you have studied it. Therefore I'm not going to depend it with a single word.
"Einstein"

* Schwarzchild Solution ($R_{\mu\nu} = 0$)

1. Spherically symmetric body with mass M
2. field static (g_{ij} are time ind.)
3. Spacetime is empty
4. Spacetime is asymptotically flat

$\Rightarrow$

$$ds^2 = -\left(1 - \frac{2MG}{c^2 r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2MG}{c^2 r}\right)} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

Reissner - Nordstöm Solution

All assumptions of Schwarzchild are taken here but the gravitating body has now charge Q.

$\Rightarrow$ the only difference is:

$$g_{00}^{RN} = g_{00}^{Sch} + \frac{Q^2}{4\pi r^2} \qquad g_{11}^{RN} = \frac{1}{g_{00}^{RN}}$$

+ Kerr solution "Rotating gravitational body $\rightarrow$ no spherical symmetry"

+ F.R.W. solution... check "The large structure of space time" for S. Hawking

Cosmology

Rules:

1. The universe is pretty much the same everywher $\Rightarrow$

Isotropy "No matter what direction you look in at some specific pt. space looks the same $\cong$ Invariace under Rotation.

& Homogeneity "The metric is the same throughout the space". $\cong$ Invariance under translation.

2. The universe is not static $\Rightarrow$

space is isotropic & homogeneous in space but not in time.

In G.R. this means that the universe can be foliated into spacelike slices such that each slice is isotropic & homogeneous

Therefore space time $= R \times \Sigma$, where R represents the time direction & Σ is a homogen-eos & isotropic (maximally symmetric) manifold.

Then the large scale metric :

$$ds^2 = -dt^2 + \alpha^2(t)\, \gamma_{\mu\nu}(u)\, du^\mu du^\nu$$

* (u_1, u_2, u_3) are coordinates on Σ ; $\gamma_{\mu\nu}$ is the maximally symmetric metric on Σ.

* As we proceed in time Σ gets bigger & thats why we multiply it by the scale factor $\alpha(t)$ that tells how big the space like slice Σ at the moment t.

* $\gamma_{\mu\nu}$ fully symmetric $\Rightarrow$ spherically symmetric

$$\Rightarrow \gamma_{\mu\nu}(u)\, du^\mu du^\nu$$

$$= e^{2\beta(\rho)} d\rho^2 + \rho^2 \underbrace{\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}_{d\Omega^2}$$

$$\Rightarrow R_{rr} = \frac{2}{\rho}\beta' \; ;$$

$$R_{\theta\theta} = e^{-2\beta}(\rho\beta' - 1) + 1$$

$$R_{\phi\phi} = \left[e^{-2\beta}(\rho\beta' - 1) + 1\right]\sin^2\theta$$

Exercise: Calculate the connection & the Ricci tensor for $\gamma_{\mu\nu}$

Exercise: Show that for fully symmetric s-space

$$R_{\mu\nu\rho\alpha} = \kappa\left(g_{\mu\rho}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\rho}\right)$$

$$\Rightarrow R_{\nu\alpha} = 2\kappa g_{\nu\alpha}$$

Comparing $R_{\mu\nu}$ with $k\,\gamma_{\mu\nu} \Rightarrow$

$$\Rightarrow \beta = -\frac{1}{2}\ln(1 - k\rho^2)$$

$$\Rightarrow ds^2 = -dt^2 + \alpha^2(t)\left[\frac{d\rho^2}{1-k\rho^2} + \rho^2 d\Omega^2\right]$$

This is the Robertson Walker metric

✩ So far we didn't use Einstein eq. It will be used to find $\alpha(t)$

✩ with $k \to \frac{k}{|k|}$; $\rho \to \sqrt{|k|}\, r$; $\alpha \to \frac{a}{\sqrt{|k|}}$ sub., let's check how the metric changes.

$$d\rho = \sqrt{|k|}\,dr \quad \& \quad (d\rho)^2 = |k|\,dr^2$$

$$ds^2 = -dt^2 + \frac{a^2}{|k|}\left[\frac{|k|\,dr^2}{1-kr^2} + |k|\,r^2 d\Omega^2\right]$$

$$\Rightarrow ds^2 = -dt^2 + a^2(t)\underbrace{\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]}_{d\sigma^2}$$

$$\Rightarrow \quad k = -1, 0, 1$$

Finding the Scale factor:

* To calculate how a(t) looks like we need to use Einstein Eq.

* Starting with geometric part:

  We need to calculate R_{ab} & R, R_{ab} is f-n of the connection $\Rightarrow$ Connections first

* $4 \times \frac{4 \times 5}{2} = 40$ independent components.

$$\Gamma^0_{11} = \frac{a\dot{a}}{1-\kappa r^2} \qquad \Gamma^0_{22} = \dot{a} a r^2 \qquad \Gamma^0_{33} = a\dot{a} r^2 \sin^2\theta$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{30} = \Gamma^3_{03} = \frac{\dot{a}}{a}$$

$$\Gamma^1_{22} = -r(1-kr^2) \qquad \Gamma^1_{33} = -r(1-\kappa r^2)\sin^2\theta$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$$

* The non zero component of the Ricci tensor

$$R_{00} = -3\frac{\ddot{a}}{a} \qquad R_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa)$$

$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1-\kappa a^2} \qquad R_{33} = r^2(a\ddot{a} + 2\dot{a}^2 + 2\kappa)\sin^2\theta$$

* The Ricci scalar is

$$R = R^{\mu}{}_{\mu} = R^{0}{}_{0} + R^{1}{}_{1} + R^{2}{}_{2} + R^{3}{}_{3}$$

$$= g^{00} R_{00} + g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33}$$

$$R = \frac{6}{a^2}\left(a\ddot{a} + \dot{a}^2 + k\right)$$

Exercise: Show that Einstein Eq. can be also written as: $R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T - \Lambda g_{\mu\nu}\right)$

* The energy-momentum tensor in rest frame

$$T_{ab} = \begin{pmatrix} \rho & 0\ 0\ 0 \\ 0 & \\ 0 & p\, g_{\mu\nu} \\ 0 & \end{pmatrix}$$

$$\Rightarrow T^{a}{}_{b} = \mathrm{diag}(-\rho, p, p, p)$$

$$\Rightarrow T = T_{\mu}{}^{\mu} = \mathrm{trace}\, T_{\mu}{}^{\nu} = -\rho + 3p$$

* Plugging into Einstien's eq.

$$R_{ab} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

00 comp. $$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p)$$

ab = $\mu\nu$ $$R_{11} = 8\pi G\left(T_{11} - \frac{1}{2} g_{11} T\right)$$

gives

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{2k}{a^2} = 4\pi G(\rho - p)$$

* The 22, 33 components give the same eq. due to isotropy.

* Sub. the 00-comp. in the 11 comp., we get rid of $\ddot{a}$, we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$$

together with the 00-comp

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

they form the "Fiedmann eqs."

* Its "educational" to calculate the 0-comp. of the conservation of energy

$$\begin{aligned}\nabla_\mu T^\mu{}_0 &= \partial_\mu T^\mu{}_0 + \Gamma^\mu{}_{\mu\delta} T^\delta{}_0 - \Gamma^\lambda{}_{\mu 0} T^\mu{}_\lambda \\ &= \partial_0 T^0{}_0 + \Gamma^\mu{}_{\mu 0} T^0{}_0 - \Gamma^0{}_{00} T^0{}_0 \\ &\quad - \Gamma^1{}_{10} T^1{}_1 - \Gamma^2{}_{20} T^2{}_2 - \Gamma^3{}_{30} T^3{}_3 \\ &= 0\end{aligned}$$

$$\Rightarrow \quad \partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p) = 0$$

* For more progress, we make use of eq. of state (a relation btwn ρ & p). Essentially all perfect fluid relevant to cosmology obey the simple eq.

$$p = \omega \rho$$

where ω is a const ind. of time, that depend on the fluid type.

Plugging in the energy conservation eq.

$$\Rightarrow \quad \frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}$$

* For $K=0 \Rightarrow d\sigma^2 = dr^2 + r^2 d\Omega^2 = dx^2 + dy^2 + dz^2$

$\Rightarrow$ Σ is flat

* For $K=+1 \Rightarrow d\sigma^2 = \frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

let $r = \sin\chi \Rightarrow dr = \cos\chi\, d\chi$

$\Rightarrow dr^2 = \cos^2\chi\, d\chi^2 = (1-\sin^2\chi)d\chi^2 = \frac{dr^2}{1-\sin^2\chi}$

$\Rightarrow d\chi^2 = \frac{dr^2}{1-r^2}$

$\Rightarrow d\sigma^2 = d\chi^2 + \sin^2\chi\, d\Omega^2$

$\Rightarrow$ Σ is closed "S^3"

* $K=-1 \Rightarrow d\sigma^2 = \frac{dr^2}{1+r^2} + r^2 d\Omega^2$

let $r = \sinh\psi \Rightarrow dr = \cosh\psi\, d\psi$

$\Rightarrow (dr)^2 = \cosh^2\psi\, d\psi^2 \Rightarrow d\psi^2 = \frac{dr^2}{1+\sinh^2}$

$\Rightarrow d\psi^2 = \frac{dr^2}{1+r^2}$

$\Rightarrow d\sigma^2 = d\psi^2 + \sinh^2\psi\, d\Omega^2$

$\Rightarrow$ Σ is open

Finally

$$ds^2 = -dt^2 + a^2(t)\,\Sigma$$

form depends on k & ω
solution = $3 \times$ # of different fluids

form depends on k
3 - solution.

"In the light of present knowledge, these achievements seem to be almost obvious, & every intelligent student grasps them w/o much trouble. Yet the year of anxious searching in the dark, with their intense longing their alternations of confidence and exhaustion and the final emergence into the light - only those who experienced this can understand it!" Einstein