Gauge Symmetry

Let us consider first classical electromagnetism

$$A^\mu = \left(\frac{\phi}{c}, \vec{A}\right) \qquad \text{where} \qquad \begin{cases} \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi, \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

Maxwell's equations are written as

$$\begin{cases} \partial_\mu F^{\mu\nu} = \mu_0 J^\nu \\ \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \end{cases}$$

In absence of a current J_μ, these can be obtained from the Lagrangian

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

Gauge transformation

This Lagrangian, as well as the Maxwell's equations of motion, are invariant if we redefine

$$A_\mu(x) \to A_\mu(x) + \frac{1}{g}\partial_\mu \omega(x) \ ,$$ where $\omega(x)$ is an arbitrary function of the position

This **gauge invariance** is crucial and fixes how the em field interacts with other fields.

Gauge invariance is introduced BECAUSE the photon is MASSLESS. In fact it **FORBIDS A MASS TERM** for A_μ:

$$\mathcal{L}^{m_A} \supset \frac{1}{2} m_A^2 A_\mu A^\mu \to \text{not invariant}$$

INTERACTION WITH MATTER

Let's introduce a complex scalar field $\varphi(x) = \dfrac{\varphi_1(x) + i\varphi_2(x)}{\sqrt{2}}$.

Its kinetic Lagrangian is $\mathcal{L}^{kin} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi = \frac{1}{2}\partial_\mu \varphi_i \partial^\mu \varphi_i - \frac{1}{2} m^2 \varphi_i^2$

The equation of motion is the Klein-Gordon equation $(\Box + m^2)\varphi(x) = 0$

$\mathcal{L}^{kin}$ is symmetric under the transformation $\varphi(x) \to e^{i\omega}\varphi(x)$, $\varphi^\dagger(x) \to e^{-i\omega}\varphi^\dagger(x)$ with $\omega = \text{const}$

This is a **global U(1) transf** (or a $SO(2)$ rotation in $(\varphi_1(x), \varphi_2(x))$)

We want this symmetry to be local: $\omega \to \omega(x)$.

$$\mathcal{L}^{kin} \to \mathcal{L}^{kin} - i\partial_\mu \omega\, \varphi^\dagger \partial^\mu \varphi + i\partial^\mu \omega (\partial_\mu \varphi^\dagger)\varphi + \partial_\mu \omega \partial^\mu \omega\, \varphi^\dagger \varphi$$

To restore the invariance we notice that $(\partial_\mu - igA_\mu)\varphi(x) \to e^{i\omega(x)} \overbrace{(\partial_\mu - igA_\mu)}^{D_\mu \text{ covariant derivative}}\varphi(x)$

$$\boxed{\mathcal{L} = (D_\mu \varphi)^\dagger (D^\mu \varphi) - m^2 \varphi^\dagger \varphi} \leftarrow \text{is invariant}$$

SPONTANEOUS BREAKING OF A SYMMETRY

A symmetry is spontaneously broken when the dynamics of the system is symmetric but the ground state is not.

For example a ferromagnet. The dynamics (Hamiltonian) of spin interactions and magnetic fields have rotational $SO(3)$ invariance. Instead the ground state has a preferred direction.

Let us consider a classical scalar field (complex) $\varphi(x) = \frac{1}{\sqrt{2}}\left(\varphi_1(x) + i\varphi_2(x)\right)$

$$\mathcal{L} = \mathcal{L}^{kin} - V(\varphi, \varphi^\dagger) \quad \Leftarrow \text{ The Lagrangian has to be Hermitian}$$

We take the special case

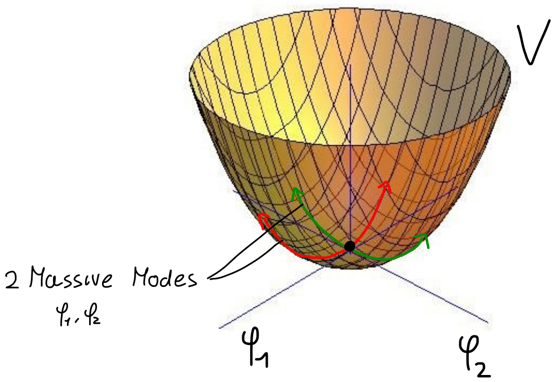
$$\mathcal{L} = (\partial_\mu \varphi^\dagger)(\partial^\mu \varphi) - V(\varphi^\dagger\varphi) = \frac{1}{2}(\partial_\mu \varphi_1)^2 + \frac{1}{2}(\partial_\mu \varphi_2)^2 - V(\varphi_1^2 + \varphi_2^2)$$

This Lagrangian is symmetric under

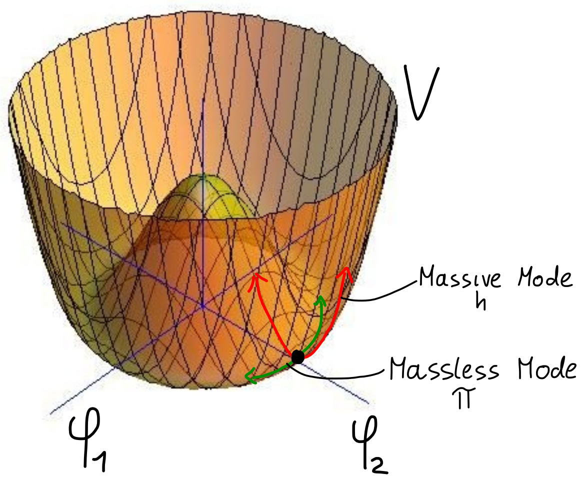
$$\begin{cases} U(1) & \varphi \to e^{i\alpha}\varphi \\ \Updownarrow & \\ SO(2) & \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix} \to \begin{pmatrix}\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}\end{cases}$$

For example $\quad V = m^2 \varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2 \qquad m^2 > 0$

The minimum of the potential is at $\varphi = 0$
Therefore the physical d.o.f are excitations around this point
In this theory they are two massive modes with same mass m



$\Rightarrow$ m is the mass of φ_1 and φ_2
λ is the self-interaction coupling



Let's now change the sign of the mass term

$$V = -m^2(\varphi_1^2+\varphi_2^2) + \lambda(\varphi_1^2+\varphi_2^2)^2$$

$m^2>0$

Now $\varphi=0$ is a local maximum
The family of minima are all the points with

$$|\varphi| = \sqrt{\frac{m^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}}$$

Now this is the ground state, so we need to expand the fields around this point to obtain the physical degrees of freedom
Let's parametrize φ as:

$$\varphi(x) = e^{i\frac{\pi(x)}{v}}\left(\frac{v+h(x)}{\sqrt{2}}\right)$$

With this parametrization

$$V = -\frac{\lambda v^4}{4} + \frac{1}{2}(2\lambda v^2)h^2 + v\lambda h^3 + \frac{\lambda}{4}h^4$$

$$m_h^2 = 2\lambda v^2$$

h Higgs boson
π Goldstone boson

Now we have 2 real dof one massive and one massless

Nambu-Goldstone theorem

Every system with a spontaneously broken continuos symmetry will have a massless mode

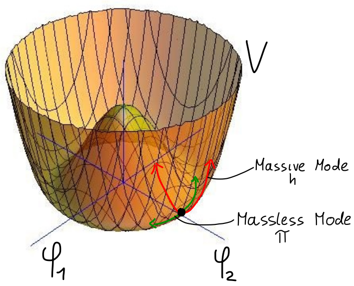
Higgs Mechanism

Let's now put together the two theories studied so far

U(1) gauge theory of a complex scalar field

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) \underbrace{+ m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2}_{-V}, \quad m^2>0$$

$$D_\mu = \partial_\mu - ig A_\mu$$



This theory has $\underset{A_\mu}{2} + \underset{\varphi}{2} = 4$ degrees of freedom

We saw before that the ground state is at $|\varphi| = \frac{v}{\sqrt{2}} \neq 0$. So we need to expand around this vacuum

$$\varphi(x) = e^{i\frac{\pi(x)}{v}}\left(\frac{v+h(x)}{\sqrt{2}}\right)$$

The potential is as before $m_h^2 = 2\lambda v^2$, $m_\pi = 0$

Let's study the kinetic term. First of all, let's use gauge invariance to remove $\pi(x)$

$$\varphi(x) \to e^{i\alpha(x)}e^{i\pi(x)/v}\frac{h+v}{\sqrt{2}} \equiv \frac{h+v}{\sqrt{2}} \quad \to \text{choose} \quad \alpha(x) = -\frac{\pi(x)}{v}$$

In this gauge

$$(D_\mu\phi)^\dagger(D^\mu\phi) = \frac{1}{2}\left[(\partial_\mu + igA_\mu)(h+v)\right]\left[(\partial^\mu - igA^\mu)(h+v)\right] =$$

$$= \frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}g^2v^2 A_\mu A^\mu\left(1+\frac{h}{v}\right)^2$$

$$m_A^2 = g^2v^2$$

We found a mass term for A_μ! (plus hAA and $hhAA$ interactions)
The massive A has 3 d.o.f: it ate the $\pi(x)$ to be its longitudinal polarization

Note: the mass of A_μ is proportional to its coupling with ϕ: g
The proportionality factor is the vacuum expectation value of ϕ: v

ELECTROWEAK SYMMETRY BREAKING

Let's consider only the electroweak gauge symmetry group (QCD doesn't play any role for this discussion) In particular, we only need to study the Higgs sector

$$G_{EW} = SU(2)_L \otimes U(1)_Y \qquad H = \begin{pmatrix} H^1 \\ H^2 \end{pmatrix} \sim 2_{1/2}$$

$$V = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2 , \quad \mu^2 > 0$$

With this potential $\langle |H| \rangle = \sqrt{\frac{\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}} \quad \Rightarrow \quad v \equiv \lambda \mu \simeq 246 \text{ GeV}$

Gauge boson masses

As we did before, we parametrize $H = \exp\left(i \frac{\pi^a}{v} \frac{\sigma^a}{2}\right) \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix}$ $\qquad \sigma^{1,2,3}$ Pauli Matrices

$$\mathcal{L} \supset (D_\mu H)^\dagger (D^\mu H) , \qquad D_\mu H = \left(\partial_\mu - i g \frac{\sigma^a}{2} W^a_\mu - i g' \frac{1}{2} B_\mu\right) H$$

We go in the Unitary gauge, where $\pi^a(x) = 0 \quad \Rightarrow \quad H = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix}$

$$(0 , 1)\, \sigma^a \sigma^b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \delta^{ab} , \qquad (0 \;\; 1)\, \sigma^a \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\delta^{a3}$$

It's a matter of algebra to show that the Higgs kinetic term become

$$(D_\mu H)^\dagger (D^\mu H) = \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{1}{2} \frac{v^2}{4} \left[g^2 (W^1_\mu)^2 + g^2 (W^2_\mu) + (g W^3_\mu - g' B_\mu)^2 \right] \left(1 + \frac{h}{v}\right)^2$$

$$= \frac{1}{2} (\partial_\mu h)^2 + \Bigg[\underbrace{\frac{g^2 v^2}{4}}_{m_W^2} W^+_\mu W^{-\mu} + \frac{1}{2} \underbrace{\frac{g^2 v^2}{4 \cos^2\theta_W}}_{m_Z^2} Z_\mu Z^\mu \Bigg] \left(1 + \frac{h}{v}\right)^2 \quad \Rightarrow \quad m_A = 0$$

Where $\quad W^\pm_\mu = \frac{1}{\sqrt{2}} (W^1_\mu \mp i W^2_\mu) , \qquad Z_\mu = \cos\theta_W W^3_\mu - \sin\theta_W B_\mu , \qquad A_\mu = \sin\theta_W W^3_\mu + \cos\theta_W B_\mu$

$\tan\theta_W = \frac{g'}{g} \quad \Rightarrow \quad \cos\theta_W = \frac{g}{(g^2 + g'^2)^{1/2}} \qquad \boxed{e = g \sin\theta_W}$ Electromagnetic coupling

Elementary fermion masses

Dirac-type fermion masses arise from $\mathcal{L} \supset -m\bar{\Psi}\Psi = -m(\bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L)$
In the SM such terms are forbidden by gauge invariance since left and right-handed fermions are in different representations of the gauge group
However we can write Yukawa interactions involving also the Higgs
Consider for example the electron

$$\mathcal{L}^{Yukawa} = y_e \underset{Y=\frac{1}{2}}{\bar{\ell}_L} \underset{Y=\frac{1}{2}}{H} \underset{Y=-1}{e_R} + y_e^* \bar{e}_R H^\dagger \ell_L \Longrightarrow \begin{cases} \text{In the Unitary gauge} \quad H = \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} \\ \mathcal{L}^{Yuk} = \underbrace{\left(\frac{y_e v}{\sqrt{2}}\right)}_{m_e}(\bar{e}_L e_R + \bar{e}_R e_L)\left(1+\frac{h}{v}\right) \end{cases}$$

Again, the electron mass is proportional to the electron-Higgs coupling and to the Higgs vacuum expectation value

The fermion which is coupled most strongly to the Higgs is the heaviest one the top

$y_t = \frac{m_t}{v}\sqrt{2} \sim 1$, $\quad y_e = \frac{m_e}{v}\sqrt{2} \sim 10^{-5}$ is the smallest one

Why do we have such a big hierarchy of Yukawas? FLAVOUR PROBLEM

Hierarchy Problem

The Standard Model is a theory with 2 scales Λ_{QCD} and v
In reality, one should add another scale the energy at which the SM ceases to be valid because some new degree of freedom at that scale needs to be considered
This scale is the cutoff of the theory Λ

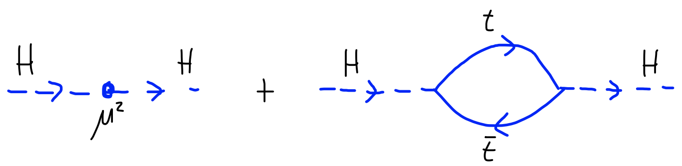
There are many hints that point to "new physics" Dark Matter, neutrino masses, strong QCD problem, gauge coupling unification and, finally, the necessity of a theory of Quantum Gravity at the Planck scale M_{Pl}.

Λ_{QCD} is a dynamically generated scale through the energy dependence of the strong coupling $g_s(E)$ It is naturally much smaller than the Planck mass.

$$\Lambda_{QCD} = \Lambda \exp\left(-\frac{8\pi^2}{b_0 g_s^2(\Lambda)}\right), \quad b_0 = 11 - \frac{2}{3} n_f$$

v is related to a scale inserted by hand in the theory $v = \mu/\lambda^{1/2}$.
In the SM there is no dynamical mechanism to explain why $v \ll \Lambda$

The problem is even deeper than this At the quantum level, fluctuations tend to push the EW scale up to the cutoff scale Λ

$\mu^2_{phys} =$  $\simeq \mu^2 - \frac{y_t^2}{16\pi^2}\Lambda^2$

Let's take for example $\Lambda \sim M_{Pl}$ The LHS is $\sim 10^4$ GeV^2 The loop correction in the RHS is of order $\sim 10^{36}$ GeV^2 Therefore the "bare" parameter μ^2 has to be tuned to one part on 10^{32} to cancel this huge contribution

Fine Tuning

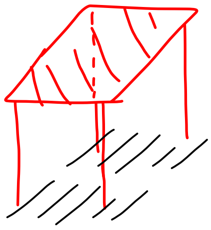
The problem is even more deep than this Any heavy degree of freedom in the high energy theory would tend to push μ^2 to that scale ← Weinberg formulation

and even MORE deep Technical naturalness ('t Hooft)
A parameter of the theory is technically natural (protected from big quantum corrections) if the theory gains a new symmetry when this parameter is set to zero

Example fermion masses are natural because the theory gains chiral symmetry when $m=0$ Analogously, a vector mass is natural because if $M_V=0$ one has gauge symmetry

Analogy (from A Pomarol)

Light fermion
$\Downarrow$
stable



Light vector
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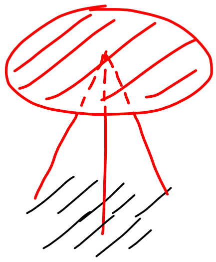
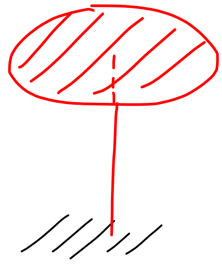
stable



Light scalar
$\Downarrow$
unstable

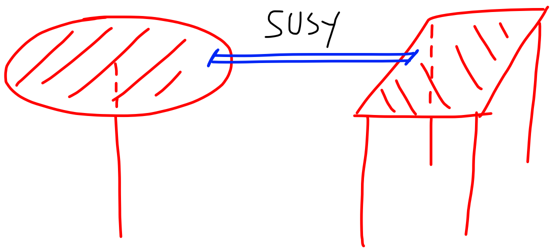


If we want our theory to be natural, the only solution conceived so far is to have new degrees of freedom at a scale $\Lambda \lesssim \left(\frac{16\pi^2}{y_t}\right)^{\frac{1}{2}} \mu \sim \text{TeV}$, without introducing new unnaturally small parameters

$$\boxed{\text{We need} \quad \Lambda_{NP} \sim \text{TeV}} \quad \Leftarrow \text{LHC Energy!!}$$

How do Beyond the Standard Model theories solve this problem?

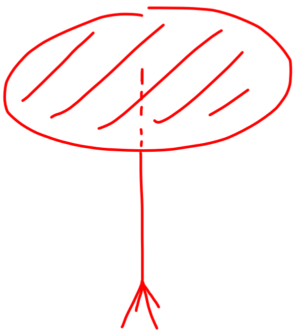
SUPERSYMMETRY extends spacetime symmetries (Poincaré group) to include also fermionic generators (approximatively, it adds a fermionic dimension)
The effect of this is that to each bosonic degree of freedom must correspond a fermionic d.o.f. and viceversa. The new particles that must be introduced are called SUPERPARTNERS
In the limit in which SUSY is unbroken, particles and antiparticles would have the same mass. Therefore, the Higgs mass is protected by the fact that the Higgsino (its spartner) is naturally light



$$\mu^2 \sim \mu_0^2 - \frac{y_t^2}{16\pi^2} \Lambda_{SUSY}^2$$

$\Rightarrow$ To be natural, the superpartners have to be near the TeV

COMPOSITE HIGGS Assume that the Higgs is a composite state of more fundamental constituents (like the pion is a composite state of quarks and antiquarks) of a NEW strongly coupled theory at the TeV $\Lambda_{CH} \sim$ TeV.



For energies $\gtrsim$ TeV the Higgs doesn't exist anymore, but we need to describe the physics in terms of the (still unknown) degrees of freedom

We expect to see resonances at the TeV scale in this case