CRs

Energetic particles hitting Earth's atmosphere from all directions

E range: 10^9 eV — 10^{20} eV $^{(*)}$

(*) Oh-my-god particle 1991 $3 \cdot 10^{20}$ eV (48 Joules, baseball with $v = 90$ km/s)

CR discovery it was known that there was some ionizing radiation in 18^{th} - 19^{th} century.

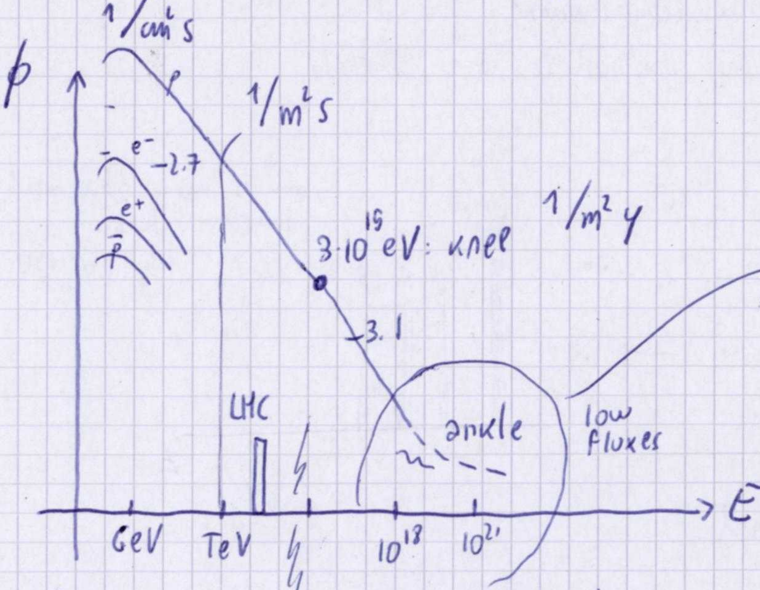
Electroscope discharge (dependent of air pressure) 18^{th} – 19^{th} cent.

- Domenico Pacini (1907-1911): discovered a lower ionization undersea
- V. Hess (1912): higher ionization at high altitude (up to 5300 m, during total eclipse)

Particle physics discoveries in CRs:
- positron '32
- muon '36 (C. Anderson)

Effects of CRs on
- cloud formation ($\rightarrow$ climate)
- life evolution? (DNA mutations)

Flux

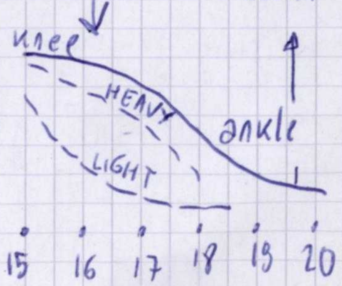


space observatories
PAMELA AMS GeV (TeV)
baloon experiments
CREAM (up to knee)

Kascade-Grande - - - ->
ICETOP (Gran Sasso)

Hires
Auger (Argentina)

$p + \gamma_{CMB} \rightarrow \Delta^+ \rightarrow$
$\rightarrow p + \pi^0$
GZK Cutoff



Origin of CRs

Zwicky & Baade 1934 proposed an argument based on energetics

$$\rho_{CR} \sim 1 \frac{eV}{cm^3} \qquad \left\{ cfr. \quad \begin{array}{l} U_B \sim 0,3\ eV/cm^3 \\ U_{\gamma, CMB\ optical} \sim 0,3\ eV/cm^3 \end{array} \right)$$

(We will see later) $\left[\begin{array}{l} \tau_{esc} \sim 6 \cdot 10^6\ y \\ V \sim \pi R^2 h \sim 4 \cdot 10^{66}\ cm^3 \end{array} \right.$

$\left[\begin{array}{l} R = 15\ kpc \\ h = 200\ pc \end{array} \right. \qquad 1pc = 3 \cdot 10^{18}\ cm$

$$L_{CRs} = \frac{V_D \rho_{CR}}{\tau_{esc}} \simeq 5 \cdot 10^{40}\ erg/s$$

1 SN ($10 M_\odot$) $\quad E_{tot} = 10^{51}\ erg \qquad L_{SNR} \simeq 10^{42}\ erg/s$

$\tau \sim 1/30\ y$

1 SN every $3 \cdot 10^9\ s$ $\qquad$ efficiency $1\% \div 10\%$

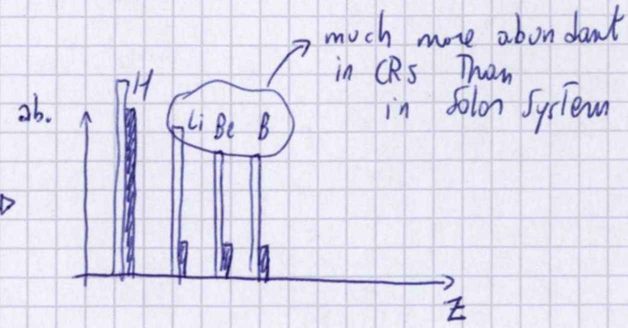
($1y = 3 \cdot 10^7\ s$)

Is this argument correct? Let's see Tomorrow (it is still a hot Topic)

Now let's come back To The confinement.
Let's prove That CRs are confined in a Halo

CR confinement

- IsoTropy of CRs
- Abundances of Li, Be, B $\longrightarrow$
- $^{10}Be/^{9}Be$



we are going To show That

$$\tau \sim 10^6 - 10^7\ y$$

③

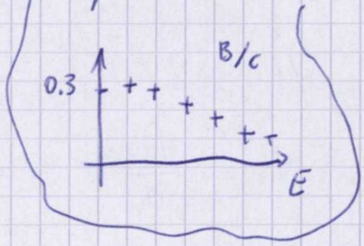
$$p \longrightarrow s + X$$
$$\begin{bmatrix} C \\ N \\ O \end{bmatrix} \quad \begin{bmatrix} Li \\ Be \\ B \end{bmatrix}$$

$$X \equiv \oint dl \, \rho_{gas}(l)$$

$$\begin{cases} dn_s = dl \, n_{gas} \, \sigma \, n_p = \dfrac{dX}{m} \sigma \, n_p \\[2ex] dn_p = - dl \, n_{gas} \, \sigma \, n_p \end{cases}$$

$$\begin{cases} \dfrac{dn_p}{dX} = - n_p \dfrac{\sigma_{tot}}{m} \quad \Rightarrow \quad n_p = n_p(0)\, e^{-\frac{\sigma_{tot} X}{m}} \\[2ex] \dfrac{dn_s}{dX} = n_p \dfrac{\sigma_{spall}}{m} \quad \Rightarrow \quad n_s = \left(n_p(0)\, e^{-\frac{\sigma_{tot} X}{m}} \right) \cdot X \dfrac{\sigma_{spall}}{m} \end{cases}$$

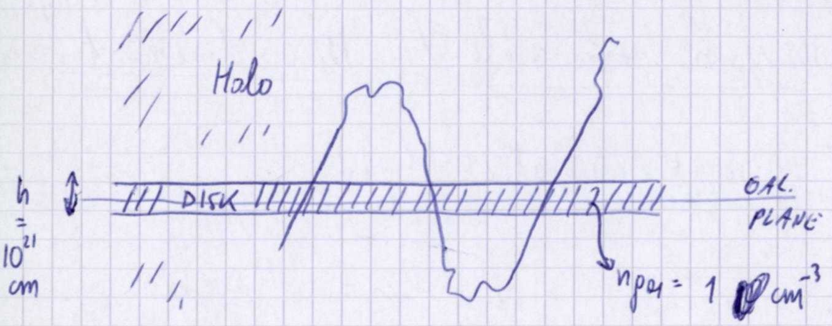
$$\left[n_s / n_p \sim X \frac{\sigma_{spall}}{m} \right]$$ $\Rightarrow$ This is why secondary / primary ratios are important



1 GeV : $\quad 0{,}3 \sim \dfrac{X}{\left(m/\sigma = 10 \, g/cm^2 \right)}$

$\Rightarrow X \sim 3 \, g/cm^2$ [GRAMMAGE]

what do we learn from this number?



$$X_{disk} = \underset{10^{-24}}{m_p} \; \underset{1}{n_{gas}} \; \underset{10^{21}}{h} \sim$$
$$\sim 10^{-3} \, g/cm^2$$
$$\Downarrow$$

CRs must cross the disk many times to accumulate the grammage

$1 \, pc = 3 \cdot 10^{18}$ cm

$$\tau = \left(\frac{X}{X_{disk}} \right) \left(\frac{h}{c} \right) = \qquad (h = 10^{21}, \; c = 3 \cdot 10^{10})$$

$$= 1{,}4 \cdot 10^{14} \, s = 5 \cdot 10^{6} \, y$$

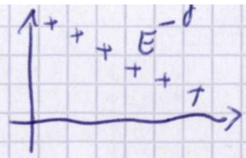
~~Larmor~~
~~Ratios argument~~
~~& energy dependence!!~~

$$\tau \propto X$$

(Ho' up To now)

Leaky-box model

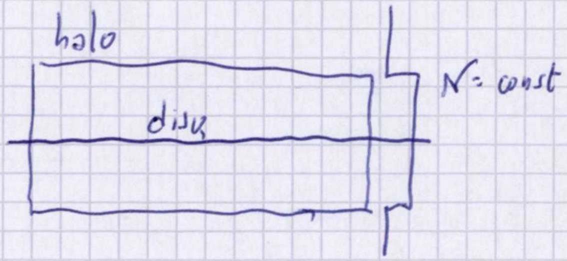
$B/C \sim X \sim \tau$



Since The B/C ratio decreases with E, The grammage (and The residence Time) also decreases $\quad X \sim \tau$

Let's sketch a simple model

$$\frac{\partial N}{\partial t} + \frac{N}{\tau(E)} = Q$$



$$Q = 0 \Rightarrow N \propto e^{-t/\tau(E)}$$

$$\text{steady-state solution} \quad Q(E) = E^{-\alpha} \Rightarrow \underbrace{N(E)}_{E^{-(\alpha+\delta)}} = \underbrace{Q(E)}_{E^{-\alpha}}\,\underbrace{\tau(E)}_{E^{-\delta}}$$

$$\begin{cases} N_p/\tau(E) = Q_p & \Rightarrow N_p = Q\,\tau(E) \\ N_s/\tau(E) = N_p\,\sigma\,\frac{\rho_{gas}}{m_p}\,v & \Rightarrow N_s = Q\,\tau^2(E)\,\sigma v\,\frac{\rho_{gas}}{m_p} \end{cases}$$

$$\frac{N_s}{N_p} = \frac{Q\tau}{Q\tau}\,\sigma v\,\frac{\rho}{m}\,\tau = \boxed{\frac{\sigma}{m}\,X(E)}$$

Which is The physical mechanism responsible for Confinement?

→ The interaction with The Turbulent magnetic field

- magnetic field

Regular + Random

$$R_L = 1\,\text{kpc}\left(\frac{E\,[10^{18}\,\text{eV}]}{B\,[\mu G]}\right)$$

- Random field is embedded in a TURBULENT MAGNETIZED PLASMA

[The confinement is governed by Larmor radius]

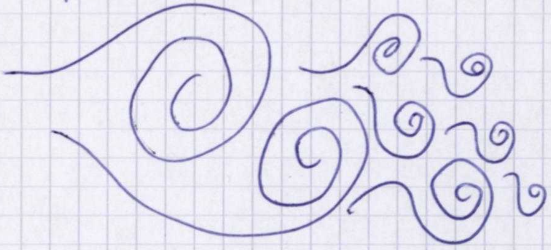
$\Rightarrow$ Galactic VS Extragalactic !!!

MHD Turbulence

Recap:

- Normal (Kolmogorov) Turbulence

a dissipation mechanism



K Energy is Transferred from large scale to small scales Through interaction of eddies

What happens in MHD plasma?

Recap

(50')

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 & \text{(Continuity)} \qquad \dfrac{d}{dt}\int_V \rho\, dV = \oint_S d\hat{S}\cdot \rho\vec{v} \\ -\dfrac{1}{c}\dfrac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} & \text{(Faraday)} \qquad (\vec{\nabla}\cdot\vec{E}=0) \quad \langle q\rangle = 0 \\ \dfrac{4\pi}{c}\vec{J} = \vec{\nabla}\times\vec{B} & \text{(Ampère-Maxwell low-f)} \qquad (\vec{\nabla}\cdot\vec{B}=0) \\ \rho\left(\dfrac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\vec{v} = \dfrac{\vec{J}\times\vec{B}}{c} - \vec{\nabla}p & \text{(Euler)} \end{cases}$$

Lorentz force on a fluid parcel

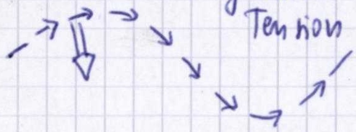
$$\frac{1}{c}\vec{J}\times\vec{B} = \frac{1}{4\pi}(\vec{\nabla}\times\vec{B})\times\vec{B} = \frac{1}{4\pi}\left\{(\vec{B}\cdot\vec{\nabla})\vec{B} - \vec{\nabla}\left(B^2/2\right)\right\}$$

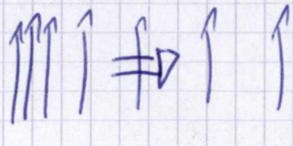
$$\left(\vec{\nabla}\left(\frac{1}{2}\vec{B}\cdot\vec{B}\right) = \vec{B}\times(\vec{\nabla}\times\vec{B}) + (\vec{B}\cdot\vec{\nabla})\vec{B}\right)$$

$$\vec{f} = \frac{1}{4\pi}(\vec{B}\cdot\vec{\nabla})\vec{B} - \vec{\nabla}\left(B^2/8\pi\right)$$

magnetic pressure

magnetic Tension



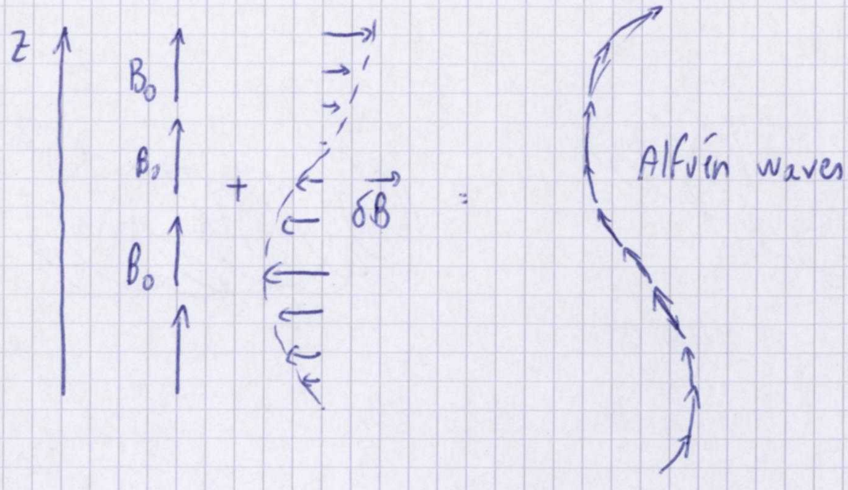


Magnetic Tension is the restoring force in Alfvén waves

MHD Turbulence: Energy is transferred from Alfvén waves with large λ to Alfvén waves with small λ

Deriving the diffusion equation

Let's consider the interaction between a CR and an Alfvén wave



$$|\delta B / B| \ll 1$$

Let's recall that in the MHD plasma we have wavepackets with random phase at each λ;

If $P(k)$ is the power associated to a length scale, $P(k) \propto k^{-\gamma}$

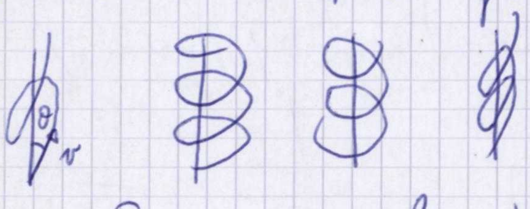
First, let's write the motion of a charged CR in the regular field

$$\frac{d\vec{p}}{dt} = q\left(E + \frac{\vec{v}}{c} \times \vec{B}\right)$$

Let's define:

$\to \Omega = \frac{qB}{mc\gamma}$ cycloTr freq

$\to \mu = \cos\theta \quad \to \theta$ = pitch angle



The motion is a spiral. $|\vec{p}|$ is conserved. μ is conserved

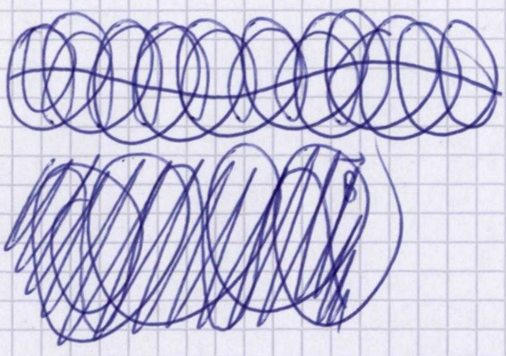
$$\begin{cases} v_x = v_\perp \cos(\Omega t + \varphi_0) \\ v_y = v_\perp \sin(\Omega t + \varphi_0) \\ v_z = v_\parallel = \text{cost} \end{cases}$$

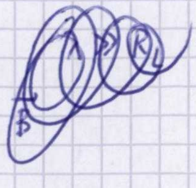
$v_\parallel = v\mu$

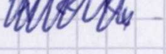
$v_\perp = v\sqrt{1-\mu^2}$

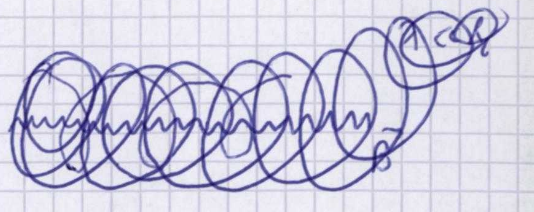
Now I will show that, if the CR interacts with a Alfvén wave, the μ is randomly changed in the interaction when there is resonance between R_{Larmor} and $\lambda_{Alfvén}$

INTUITIVE PICTURE:

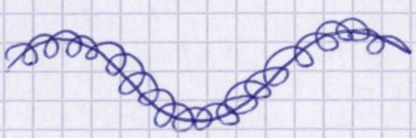




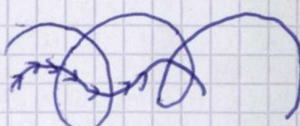




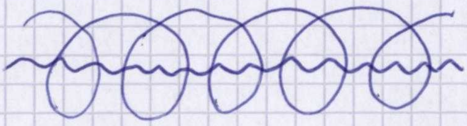
1) $\lambda \gg R_L$



3) $\lambda \approx R_L$



2) $\lambda \ll R_L$



Lorentz eq along z

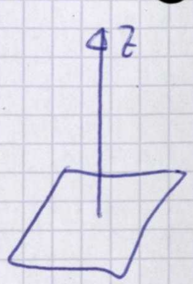
$$m\ddot{z} = m \frac{d}{dt}(v_z) = p\frac{d\mu}{dt}$$

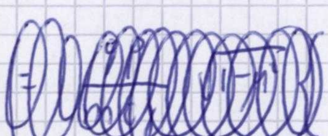
$$\vec{B} = B_0 \hat{z} + \delta\vec{B}$$

$$\begin{cases} \delta B_x = |\delta B_k| \sin(kz + \varphi) \\ \delta B_y = -|\delta B_k| \cos(kz+\varphi) \end{cases}$$

$$p\frac{d\mu}{dt} = q\frac{\vec{v}}{c} \times \vec{B} =$$

- Regular field along $\hat{z}$
- Random-phase Alfvén wave travelling along $\hat{z}$ polarized on the xy plane





$$= \frac{q}{c} v_\perp^0 \left\{ \cos\Omega t \, \delta B_y - \sin\Omega t \, \delta B_x \right\}$$

$$v_{\perp_0} = v\sqrt{1-\mu^2}$$

$$= \underbrace{\frac{qp}{mc\gamma}}_{\Omega/B_0} \sqrt{1-\mu^2} \left\{ \cos\Omega t \, \delta B_y - \sin\Omega t \, \delta B_x \right\}$$

$$p = mc\gamma v$$

$$\left(v_\perp = \frac{p}{m\gamma}\sqrt{1-\mu^2} \right)$$

$$\frac{d\mu}{dt} = \frac{q\sqrt{1-\mu^2}}{mc\gamma} |\delta B_k| \left\{ \cos(\Omega t) \cos(kz+\varphi) + \right.$$
$$\left. + \sin(\Omega t) \sin(kz+\varphi) \right\}$$

$$(z = v\mu t)$$

$$\begin{cases} \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \end{cases}$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

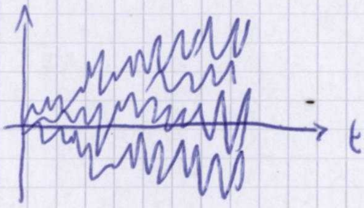
Let's recall that φ is random

$\Delta\mu$ is random

μ varies with a random walk

$\langle \mu \rangle = 0$ → The cosine of a random phase is zero

$\langle \mu^2 \rangle = 0$



Let's compute the average displacement square

(According to Kubo '57

$$D_{\mu\mu} \equiv \lim_{t\to\infty} \left\langle \frac{(\Delta\mu)^2}{2\Delta t} \right\rangle_\varphi)$$

$$\langle(\Delta\mu\,\Delta\mu)\rangle = \left\langle \frac{q^2(1-\mu^2)}{m^2c^2\gamma^2} |\delta B_k|^2 \int dt' \int dt'' \cos\{(\Omega - v\mu)t' - \varphi\}\right.$$

$$\left. \cos\{(\Omega - v\mu)t'' - \varphi\} \right\rangle_\varphi \cdot //$$

↓ Prostaferesi

$$\left\langle \int dt' \int dt'' \cos(\;)\cos(\;) \right\rangle = \left\langle \int dt' \int dt'' \frac{1}{2} \left\{ \cos\left[\cancel{(\Omega - v\mu)(t'+t'') - 2\varphi}\right]^{\nearrow 0} + \right.\right.$$

$$+ \cos\left[(\Omega - v\mu)(t' - t'')\right] \Big\rangle_\psi$$

$$\left\langle \int d\tilde{t} \int d\tilde{\tilde{t}} \; \cos\left[(\Omega - v\mu)\tilde{\tilde{t}}\right] \right\rangle_\psi$$

$$\int d\tilde{t} = \Delta t$$

$$\int dt \cos\left[(\Omega - v\mu)t\right] = \frac{2\pi}{\mu v} \, \delta\left(k - \frac{\Omega}{\mu v}\right)$$

$$\left(\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \delta(\vartheta) = \frac{1}{2\pi}\int dt \, e^{-i\vartheta t} \right)$$

$R_L \sim \lambda$

$$\Rightarrow \boxed{\langle \Delta\mu \Delta\mu \rangle = \frac{q^2(1-\mu^2)}{m^2 c^2 \gamma^2} |\delta B_k|^2 \, \Delta t \cdot \frac{2\pi}{\mu v} \, \delta\Big(\underbrace{k}_{1/\lambda} - \underbrace{\frac{\Omega}{\mu v}}_{1/R_L}\Big)}$$

~~ [COMMENTS]

Generalization $\qquad \dfrac{q}{mc\gamma} = \Omega / B_0$

$$\left\langle \frac{\Delta\mu \Delta\mu}{\Delta t} \right\rangle = \pi (1 - \mu^2) \, \Omega \, k_{res} \int dk \left(\frac{\delta^2 B_k}{4\pi}\right) \frac{\delta(k - k_{res})}{\left(B_0^2 / 8\pi\right)}$$

$$\propto G(k_{res}) \equiv \frac{\delta B^2}{B^2}(k_{res}) = \frac{P(k)}{B^2}$$

It can be shown that the scattering rate is $\qquad \mu = \cos\theta$, $\Delta\mu = \sin\theta \, \Delta\theta$

$$\nu = \left\langle \frac{\Delta\theta \, \Delta\theta}{\Delta t} \right\rangle \sim \frac{k P(k)}{B_0^2} \, \Omega$$

$$\tau \sim \frac{1}{\nu} \sim \Omega^{-1} \left(\frac{P(k)}{B_0^2 / 8\pi} \right)^{-1}$$

Time required in order to get $\delta\theta \sim 1$

From dim. analysis $\frac{\partial f}{\partial t} + D\frac{\partial f}{\partial x^2} = Q$ $D = \langle\frac{(\Delta x)^2}{\Delta t}\rangle = \langle \Delta x \frac{\Delta x}{\Delta t}\rangle$

$$D_{xx}(p) = \frac{1}{3} v \left(v\tau\right) \sim \frac{1}{3} v^2 \Omega^{-1} \left(\frac{k P(k)}{B_0^2/8\pi}\right)^{-1}$$

$$= \frac{1}{3} \frac{R_L v}{\mathcal{F}}$$

$$\mathcal{F} \propto \frac{k P(k)}{\left(B_0^2/8\pi\right)}$$

$P(k) \uparrow \Rightarrow D_{\parallel} \downarrow$ $\boxed{\frac{\partial f}{\partial t} + \frac{\partial}{\partial z} D_{\parallel} \frac{\partial}{\partial z} f + Q = 0}$

$D \approx 10^{29}$ cm²/s $\Longleftrightarrow$ $\delta B/B \sim 10^{-4}$

$\boxed{D_{\parallel} \text{ and } D_{\perp}}$ — The diffusion coefficient determined so far is parallel to the regular magnetic field; $D_{\perp} \ll D_{\perp}$ in QLT

The general equation is

$$\boxed{\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i} D_{ij} \frac{\partial}{\partial x_j} f + Q = 0}$$

$$D_{ij} = (D_{\parallel} - D_{\perp}) b_i b_j + D_{\perp} \delta_{ij}$$

if $\vec{b} = (1,0,0)$ $D_{ij} = \begin{pmatrix} D_{\parallel} & 0 & 0 \\ 0 & D_{\perp} & 0 \\ 0 & 0 & D_{\perp} \end{pmatrix}$

If turbulence increases, $D_{\parallel} \uparrow$ and $D_{\perp} \downarrow$ (intuitive picture) $\Rightarrow$ scalar equation

Solutions of the diff equation (scalar, position-independent)

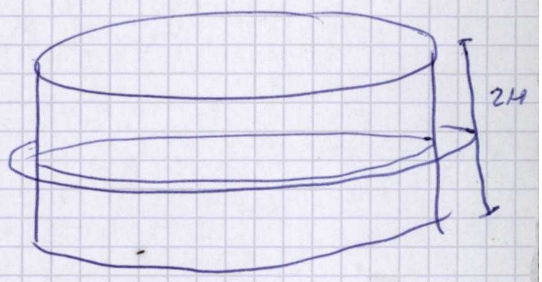
$$\frac{\partial}{\partial z}\left\{ D(E)\,\frac{\partial}{\partial z} n \right\} = -\,\delta(z)\,\frac{\overset{1/E}{N(E)}\;\overset{1/T}{R}}{\pi R_d^2}$$

$$Q = \underset{\frac{\delta}{\pi R^2}}{f(x,y,z)}\;\underset{\frac{1}{E}}{N(E)}\;\underset{\frac{1}{T}}{R}$$

$$\int dE \int dV \int dt\; Q = [N_{CR}]$$

$$\int^{R_d} \frac{N(E)\,R}{\pi R_d^2}\,\delta(z)\,2\pi r\,dr\,dz = \frac{1}{\cancel{\pi R_d^2}}\,\cancel{\pi R_d^2}\,N(E)\,R$$

$$\int dt \int dE\; N(E)\,R = N$$



$$D(E)\,\frac{\partial^2 n}{\partial z^2} = -\,\delta(z)\,\frac{N(E)\,R}{\pi d^2}$$

$$\Downarrow$$

τ = confinement time

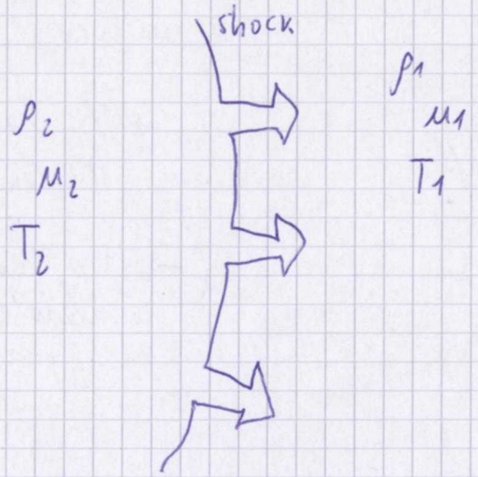
$$N_{CR}(E) = \frac{N(E)\,R}{2H\,\pi\,R_d^2}\;\frac{H^2}{D(E)}$$

Green Function

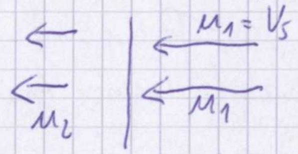
$$g = \frac{N(E)}{(4\pi D t)^{3/2}}\; e^{-|\vec{r}|^2/4Dt}$$

Acceleration

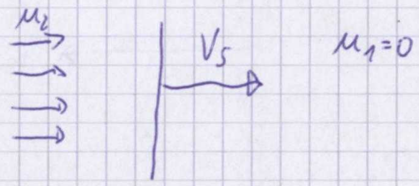
1) Shock wave equations



in The shock reference frame



in The lab reference frame



Rankine - Hugoniot conditions

$$\begin{cases} \rho_1 u_1 = \rho_2 u_2 \\ \rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \\ \frac{1}{2} u_1^2 + \varepsilon_1 + \frac{P_1}{\rho_1} = \frac{1}{2} u_2^2 + \varepsilon_2 + \frac{P_2}{\rho_2} \end{cases}$$

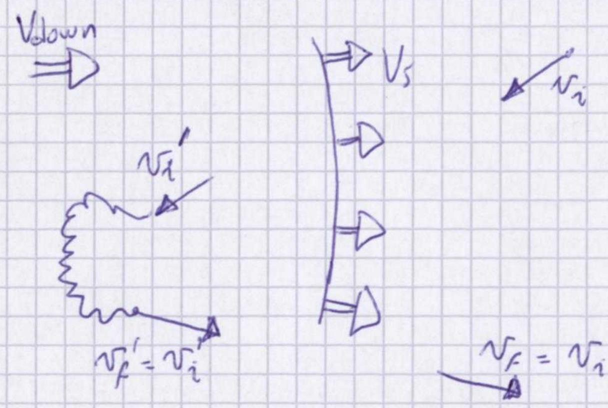
$$M \equiv \frac{u_1}{c_s} = \left(\frac{\rho_1 u_1^2}{\gamma P_1} \right)^{1/2}$$

One can solve This set of equations

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} \approx \frac{\gamma + 1}{\gamma - 1} = 4 \quad \text{if } M_1 \gg 1 \quad \text{(strong shock)}$$

we will use This concept later.

- Non-relativistic acceleration



$\vec{v}_f = v_i + V_{down}$

we need:

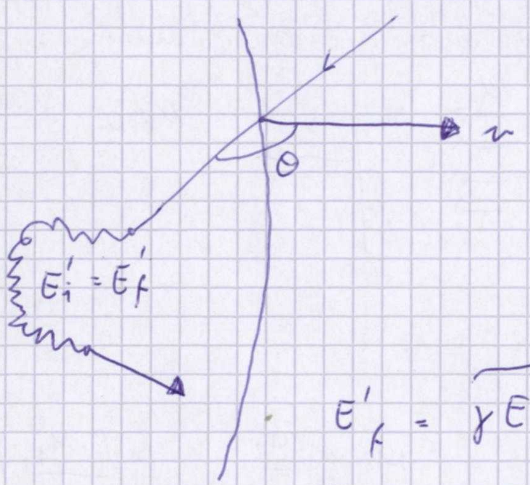
→ Turbulence

→ Luck

→ A shock providing the energy

(→ How can a B field accelerate the particles?)

- Relativistic acceleration



E_i

$E_i' = \gamma E_i (1 - \beta \cos\theta_1)$

in the ~~[illegible]~~ downstr. reference frame

$\beta = \frac{u_1 - u_2}{c}$

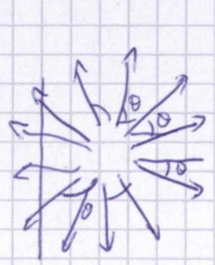
$$E_f' = \overbrace{\gamma E_i (1 - \beta \cos\theta_1)}^{E_i'}$$

$$E_f = \gamma E_f' (1 + \beta \cos\theta_2')$$

$$\frac{\Delta E}{E} = \frac{\gamma^2 E_i (1 - \beta\cos\theta_1)(1 + \beta\cos\theta_2') - E_1}{E_1} = \frac{1 - \beta\cos\theta_1 + \beta\cos\theta_2 - \beta^2\cos\theta_1\cos\theta_2}{1-\beta^2}$$

-1

$\langle \frac{\Delta E}{E} \rangle = ?$ it is a stochastic process ... we have to compute the average



$\mu = \cos\theta$ The projection of an isotropic flux is

$$\frac{dn}{d\mu} = \begin{cases} 2\mu & \mu < 0 \\ 0 & \mu > 0 \end{cases}$$

(the opposite on the other side)

$$\left\langle \frac{\Delta E}{E} \right\rangle = \int_{-1}^{0} 2\mu_1 \, d\mu_1 \int_{0}^{1} 2\mu_2 \, d\mu_2 \left\{ \gamma^2 (1-\beta\mu_1)(1+\beta\mu_2) - 1 \right\} =$$

1) $\displaystyle\int_{-1}^{0} 2\mu \, d\mu \, (1-\beta\mu) = 2\frac{\mu^2}{2}\Big|_{-1}^{0} - 2\beta\frac{\mu^3}{3}\Big|_{-1}^{0} =$

$\displaystyle = 1 + \frac{2\beta}{3}$

2) $\displaystyle\int_{0}^{1} 2\mu \, d\mu \, (1+\beta\mu) = 1 + \frac{2\beta}{3}$

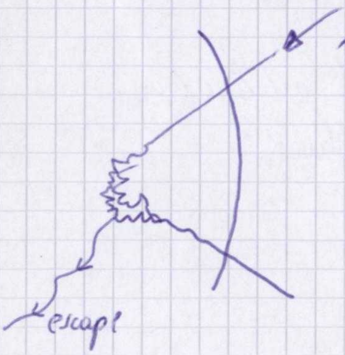
$$\left\langle \frac{\Delta E}{E} \right\rangle = \cancel{1} + \frac{2}{3}\beta + \frac{2}{3}\beta + O(\beta^2) - \cancel{1} = \frac{4}{3}\beta$$

linear in $\beta \equiv \dfrac{u_1 - u_2}{c}$

First order Fermi mechanism

~ ~ ~

What about the Energy Spectrum?



$$\left\langle \frac{\Delta E}{E} \right\rangle = \frac{4}{3}\beta \equiv \xi$$

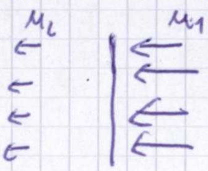
$E_n = E_0 (1+\xi)^n$ after n iterations

$\displaystyle f(\geq E_n) = \sum_{m=n}^{\infty} (1 - p_{esc})^m$ Fraction of particles with $E > E_n$

Fraction

$\displaystyle = \frac{(1-p_{esc})^n}{p_{esc}}$

(at each iteration, there is a probability p_{esc} to escape



$p_{escape} = 4\dfrac{u_2}{c}$ (*)

$$\begin{cases} \ln\left(\dfrac{E_k}{E_0}\right) = k \ln\left(1 + \dfrac{4}{3}\dfrac{u_1 - u_2}{c}\right) \\[2ex] \ln\left(\dfrac{N_k}{N_0}\right) = k \ln\left(1 - 4\dfrac{u_2}{c}\right) \end{cases}$$

$$\begin{cases} \ln\left(\dfrac{E_k}{E_0}\right) = k \ln\Big(\underbrace{1+\dfrac{4}{3}\dfrac{u_1-u_2}{c}}_{\approx \frac{4}{3}\frac{u_1-u_2}{c}}\Big) \Rightarrow k = \ln\left(\dfrac{E_k}{E_0}\right) \Big/ \left(\dfrac{4}{3}\dfrac{u_2-u_1}{c}\right) \\ \ln\left(\dfrac{N_k}{N_0}\right) = k \ln\Big(\underbrace{1-4u_2/c}_{\approx 4\frac{u_2}{c}}\Big) \end{cases}$$

$$e^{\ln\left(\frac{N_k}{N_0}\right)} = e^{\ln\left(\frac{E_k}{E_0}\right) \boxed{\left(-4\frac{u_2}{c}\right)\frac{1}{\frac{4}{3}\frac{u_2-u_1}{c}}}}$$

$$\frac{N_k}{N_0} = \left(\frac{E_k}{E_0}\right)^{-\frac{4u_2}{\frac{4}{3}(u_1-u_2)}} \quad -\gamma$$

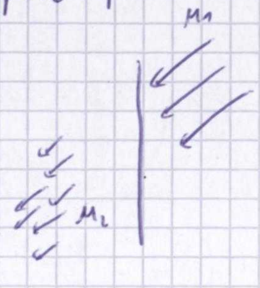
$$-\gamma = -\frac{3u_2}{u_1-u_2} = -\frac{3}{r-1} \qquad r \equiv \frac{u_1}{u_2} = 4 \text{ for strong shocks}$$

$$f(>E) \propto (E)^{-\gamma} \propto E^{-1}$$ slope of the integrated spectrum

$$\boxed{\frac{dN}{dE} \propto E^{-2}}$$ standard prediction of CR acceleration for strong shock

differential spectrum

(*) Pesc proof



$$P_{return} = \frac{\varphi_{out}}{\varphi_{in}} = \frac{\int_{-u_2}^{1} d\mu\, f_0(u_2+\mu) = \frac{1}{2}(1+u_2)^2}{\int_{-1}^{-u_2} d\mu\, f_0(u_2+\mu) = \frac{1}{2}(1-u_2)^2}$$

$$\approx 1 - 4u_2$$