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Petnica Lectures of Semiclassical Black Holes

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Abstract: A brief introduction to classical and semiclassical properties of black holes.

Contents

1. Orientation

1.1 Simple and yet Complex!

A black hole is at once the most simple and the most complex object.

It is the most simple in that it is completely specified by its mass, spin, and charge. This remarkable fact is a consequence of a the so called 'No Hair Theorem'. For an astrophysical object like the earth, the gravitational field around it depends not only on its mass but also on how the mass is distributed and on the details of the oblate-ness of the earth and on the shapes of the valleys and mountains. Not so for a black hole. Once a star collapses to form a black hole, the gravitational field around it forgets all details about the star that disappears behind the even horizon except for its mass, spin, and charge. In this respect, a black hole is very much like a structure-less elementary particle such as an electron.

And yet it is the most complex in that it possesses a huge entropy. In fact the entropy of a solar mass black hole is enormously bigger than the thermal entropy of the star that might have collapsed to form it. Entropy gives an account of the number of microscopic states of a system. Hence, the entropy of a black hole signifies an incredibly complex microstructure. In this respect, a black hole is very unlike an elementary particle.

Understanding the simplicity of a black hole falls in the realm of classical gravity. By the early seventies, full fifty years after Schwarzschild, a reasonably complete understanding of gravitational collapse and of the properties of an event horizon was achieved within classical general relativity. The final formulation began with the singularity theorems of Penrose, area theorems of Hawking and culminated in the laws of black hole mechanics.

Understanding the complex microstructure of a black hole implied by its entropy falls in the realm of quantum gravity and is the topic of present lectures. Recent developments have made it clear that a black hole is 'simple' not because it is like an elementary particle, but rather because it is like a statistical ensemble. An ensemble is also specified by a few a conserved quantum numbers such as energy, spin, and charge. The simplicity of a black hole is no different than the simplicity that characterizes a thermal ensemble.

1.2 Historical Aside

Apart from its physical significance, the entropy of a black hole makes for a fascinating study in the history of science. It is one of the very rare examples where a scientific idea has gestated and evolved over several decades into an important conceptual and quantitative tool almost entirely on the strength of theoretical considerations. That we can proceed so far with any confidence at all with very little guidance from experiment is indicative of the robustness of the basic tenets of physics. It is therefore worthwhile to place black holes and their entropy in a broader context before coming to the more recent results pertaining to the quantum aspects of black holes within string theory.

A black hole is now so much a part of our vocabulary that it can be difficult to appreciate the initial intellectual opposition to the idea of 'gravitational collapse' of a star and of a 'black hole' of nothingness in spacetime by several leading physicists, including Einstein himself.

To quote the relativist Werner Israel ,

"*There is a curious parallel between the histories of black holes and continental drift. Evidence for both was already non-ignorable by 1916, but both ideas were stopped in their tracks for half a century by a resistance bordering on the irrational.*"

On January 16, 1916, barely two months after Einstein had published the final form of his field equations for gravitation [1], he presented a paper to the Prussian Academy on behalf of Karl Schwarzschild [2], who was then fighting a war on the Russian front. Schwarzschild had found a spherically symmetric, static and exact solution of the full nonlinear equations of Einstein without any matter present.

The Schwarzschild solution was immediately accepted as the correct description within general relativity of the gravitational field outside a spherical mass. It would be the correct approximate description of the field around a star such as our sun. But something much more bizzare was implied by the solution. For an object of mass M, the solution appeared to become singular at a radius $R = 2GM/c^2$. For our sun, for example, this radius, now known as the Schwarzschild radius, would be about three kilometers. Now, as long the physical radius of the sun is bigger than three kilometers, the 'Schwarzschild's singularity' is of no concern because inside the sun the Schwarzschild solution is not applicable as there is matter present. But what if the entire mass of the sun was concentrated in a sphere of radius smaller than three kilometers? One would then have to face up to this singularity.

Einstein's reaction to the 'Schwarzschild singularity' was to seek arguments that would make such a singularity inadmissible. Clearly, he believed, a physical theory could not tolerate such singularities. This drove his to write as late as 1939, in a published paper,

"*The essential result of this investigation is a clear understanding as to why the 'Schwarzschild singularities' do not exist in physical reality.*"

This conclusion was however based on an incorrect argument. Einstein was not alone in this rejection of the unpalatable idea of a total gravitational collapse of a physical system. In the same year, in an astronomy conference in Paris, Eddington, one of the leading astronomers of the time, rubbished the work of Chandrasekhar who had concluded from his study of white dwarfs, a work that was to earn him the Nobel prize later, that a large enough star could collapse.

It is interesting that Einstein's paper on the inadmissibility of the Schwarzschild singularity appeared only two months before Oppenheimer and Snyder published their definitive work on stellar collapse with an abstract that read,

"*When all thermonuclear sources of energy are exhausted, a sufficiently heavy star will collapse.*"

Once a sufficiently big star ran out of its nuclear fuel, then there was nothing to stop the inexorable inward pull of gravity. The possibility of stellar collapse meant that a star could be compressed in a region smaller than its Schwarzschild radius and the 'Schwarzschild singularity' could no longer be wished away as Einstein had desired. Indeed it was essential to understand what it means to understand the final state of the star.

It is thus useful to keep in mind what seems now like a mere change of coordinates was at one point a matter of raging intellectual debate.

1.3 Sources

A good introductory textbook on general relativity from a modern perspective see [3]. For a more detailed treatment [4] which has become a standard reference among relativists, and [5], though a bit dated, remains a classic for various aspects of general relativity. For quantum field theory in curved spacetime see [6]. A simpler derivation can be found in [7]. The classic paper of Hawking [8] is of course worth reading in original.

2. Classical Black Holes

To understand the relevant parameters and the geometry of black holes, let us first consider the Einstein-Maxwell theory described by the action

$$\frac{1}{16\pi G}\int R\sqrt{g}d^4x - \frac{1}{16\pi}\int F^2\sqrt{g}d^4x, \tag{2.1}$$

where G is Newton's constant, $F_{\mu\nu}$ is the electro-magnetic field strength, R is the Ricci scalar of the metric $g_{\mu\nu}$. In our conventions, the indices μ, ν take values $0, 1, 2, 3$ and the metric has signature $(-, +, +, +)$.

2.1 Schwarzschild Metric

Consider the Schwarzschild metric which is a spherically symmetric, static solution of the vacuum Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} = 0$ that follow from (2.1) when no electromagnetic fields are excited. This metric is expected to describe the spacetime outside a gravitationally collapsed non-spinning star with zero charge. The solution for the line element is given by

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2,$$

where t is the time, r is the radial coordinate, and Ω is the solid angle on a 2-sphere. This metric appears to be singular at $r = 2GM$ because some of its components vanish or diverge, $g_{00} \to \infty$ and $g_{rr} \to \infty$. As is well known, this is not a real singularity. This is because the gravitational tidal forces are finite or in other words, components of Riemann tensor are finite in orthonormal coordinates. To better understand the nature of this apparent singularity, let us examine the geometry more closely near $r = 2GM$. The surface $r = 2GM$ is called the 'event horizon' of the Schwarzschild solution. Much of the interesting physics having to do with the quantum properties of black holes comes from the region near the event horizon.

To focus on the near horizon geometry in the region $(r - 2GM) \ll 2GM$, let us define $(r - 2GM) = \xi$, so that when $r \to 2GM$ we have $\xi \to 0$. The metric then takes the form

$$ds^2 = -\frac{\xi}{2GM}dt^2 + \frac{2GM}{\xi}(d\xi)^2 + (2GM)^2 d\Omega^2, \tag{2.2}$$

up to corrections that are of order $(\frac{1}{2GM})$. Introducing a new coordinate ρ,

$$\rho^2 = (8GM)\xi \quad \text{so that} \quad d\xi^2 \frac{2GM}{\xi} = d\rho^2,$$

the metric takes the form

$$ds^2 = -\frac{\rho^2}{16G^2M^2}dt^2 + d\rho^2 + (2GM)^2 d\Omega^2. \tag{2.3}$$

From the form of the metric it is clear that ρ measures the geodesic radial distance. Note that the geometry factorizes. One factor is a 2-sphere of radius $2GM$ and the other is the (ρ, t) space

$$ds_2^2 = -\frac{\rho^2}{16G^2M^2}dt^2 + d\rho^2. \tag{2.4}$$

We now show that this $1 + 1$ dimensional spacetime is just a flat Minkowski space written in funny coordinates called the Rindler coordinates.

2.2 Rindler Coordinates

To understand Rindler coordinates and their relation to the near horizon geometry of the black hole, let us start with $1 + 1$ Minkowski space with the usual flat Minkowski metric,

$$ds^2 = -dT^2 + dX^2. \tag{2.5}$$

In light-cone coordinates,

$$U = (T - X) \quad V = (T + X), \tag{2.6}$$

the line element takes the form

$$ds^2 = -dU\ dV. \tag{2.7}$$

Now we make a coordinate change

$$U = -\frac{1}{\kappa} e^{-a\kappa u}, \quad V \frac{1}{\kappa} e^{\kappa v}, \tag{2.8}$$

to introduce the Rindler coordinates (u, v). In these coordinates the line element takes the form

$$ds^2 = -dU\ dV = -e^{\kappa(v-u)} du\ dv. \tag{2.9}$$

Using further coordinate changes

$$u = (t - x), \quad v = (t + x), \quad \rho = \frac{1}{\kappa} e^{\kappa x}, \tag{2.10}$$

we can write the line element as

$$ds^2 = e^{2\kappa x}(-dt^2 + dx^2) = -\rho^2 \kappa^2 dt^2 + d\rho^2. \tag{2.11}$$

Comparing (2.4) with this Rindler metric, we see that the (ρ, t) factor of the Schwarzschild solution near $r \ \sim 2GM$ looks precisely like Rindler spacetime with metric

$$ds^2 - \rho^2 \kappa^2\ dt^2 + d\rho^2 \tag{2.12}$$

with the identification

$$\kappa = \frac{1}{4GM}.$$

This parameter κ is called the surface gravity of the black hole. For the Schwarzschild solution, one can think of it heuristically as the Newtonian acceleration GM/r_H^2 at the horizon radius $r_H = 2GM$. Both these parameters–the surface gravity κ and the horizon radius r_H play an important role in the thermodynamics of black hole.

This analysis demonstrates that the Schwarzschild spacetime near $r = 2GM$ is not singular at all. After all it looks exactly like flat Minkowski space times a sphere of radius $2GM$. So the curvatures are inverse powers of the radius of curvature $2GM$ and hence are small for large $2GM$.

2.3 Kruskal Extension

One important fact to note about the Rindler metric is that the coordinates u, v do not cover all of Minkowski space because even when the vary over the full range

$$-\infty < u < \infty, \quad -\infty < v < \infty$$

the Minkowski coordinate vary only over the quadrant

$$-\infty < U \leq 0, \quad 0 \leq V < \infty. \tag{2.13}$$

If we had written the flat metric in these 'bad', 'Rindler-like' coordinates, we would find a fake singularity at $\rho = 0$ where the metric appears to become singular. But we can discover the 'good', Minkowski-like coordinates U and V and extend them to run from $-\infty$ to ∞ to see the entire spacetime.

Since the Schwarzschild solution in the usual (r, t) Schwarzschild coordinates near $r = 2GM$ looks exactly like Minkowski space in Rindler coordinates, it suggests that we must extend it in properly chosen 'good' coordinates. As we have seen, the 'good' coordinates near $r = 2GM$ are related to the Schwarzschild coordinates in exactly the same way as the Minkowski coordinates are related the Rindler coordinates.

In fact one can choose 'good' coordinates over the entire Schwarzschild spacetime. These 'good' coordinates are called the Kruskal coordinates. To obtain the Kruskal coordinates, first introduce the 'tortoise coordinate'

$$r^* = r + 2GM \log\left(\frac{r - 2GM}{2GM}\right). \tag{2.14}$$

In the (r^*, t) coordinates, the metric is conformally flat, *i.e.*, flat up to rescaling

$$ds^2 = (1 - \frac{2GM}{r})(-dt^2 + dr^{*2}). \tag{2.15}$$

Near the horizon the coordinate r^* is similar to the coordinate x in (2.11) and hence $u = t - r^*$ and $v = t + r^*$ are like the Rindler (u, v) coordinates. This suggests that we define U, V coordinates as in (2.8) with $\kappa = 1/4GM$. In these coordinates the metric takes the form

$$ds^2 = -e^{-(v-u)\kappa} dU\ dV = -\frac{2GM}{r} e^{-r/2GM} dU\ dV \tag{2.16}$$

We now see that the Schwarzschild coordinates cover only a part of spacetime because they cover only a part of the range of the Kruskal coordinates. To see the entire spacetime, we must extend the Kruskal coordinates to run from $-\infty$ to ∞. This extension of the Schwarzschild solution is known as the Kruskal extension.

Note that now the metric is perfectly regular at $r = 2GM$ which is the surface $UV = 0$ and there is no singularity there. There is, however, a real singularity at $r = 0$ which cannot be removed by a coordinate change because physical tidal forces become infinite. Spacetime stops at $r = 0$ and at present we do not know how to describe physics near this region.

2.4 Event Horizon

We have seen that $r = 2GM$ is not a real singularity but a mere coordinate singularity which can be removed by a proper choice of coordinates. Thus, locally there is nothing special about the surface $r = 2GM$. However, globally, in terms of the causal structure of spacetime, it is a special surface and is called the 'event horizon'. An event horizon is a boundary of region in spacetime from behind which no causal signals can reach the observers sitting far away at infinity.

To see the causal structure of the event horizon, note that in the metric (2.11) near the horizon, the constant radius surfaces are determined by

$$\rho^2 = \frac{1}{\kappa^2} e^{2\kappa x} = \frac{1}{\kappa^2} e^{\kappa u} e^{-\kappa v} = -UV = \text{constant} \tag{2.17}$$

These surfaces are thus hyperbolas. The Schwarzschild metric is such that at $r \gg 2GM$ and observer who wants to remain at a fixed radial distance $r =$ constant is almost like an inertial, freely falling observers in flat space. Her trajectory is time-like and is a straight line going upwards on a spacetime diagram. Near $r = 2GM$, on the other hand, the constant r lines are hyperbolas which are the trajectories of observers in uniform acceleration.

To understand the trajectories of observers at radius $r > 2GM$, note that to stay at a fixed radial distance r from a black hole, the observer must boost the rockets to overcome gravity. Far away, the required acceleration is negligible and the observers are almost freely falling. But near $r = 2GM$ the acceleration is substantial and the observers are not freely falling. In fact at $r = 2GM$, these trajectories are light like. This means that a fiducial observer who wishes to stay at $r = 2GM$ has to move at the speed of light with respect to the freely falling observer. This can be achieved only with infinitely large acceleration. This unphysical acceleration is the origin of the coordinate singularity of the Schwarzschild coordinate system.

In summary, the surface defined by $r =$ contant is timelike for $r > 2GM$, spacelike for $r < 2GM$, and light-like or null at $r = 2GM$.

In Kruskal coordinates, at $r = 2GM$, we have $UV = 0$ which can be satisfied in two ways. Either $V = 0$, which defines the 'future event horizon', or $U = 0$, which defines the 'past event horizon'. The future event horizon is a one-way surface that signals can be sent into but cannot come out of. The region bounded by the event horizon is then a black hole. It is literally a hole in spacetime which is black because no light can come out of it. Heuristically, a black hole is black because even light cannot escape its strong gravitation pull. Our analysis of the metric makes this notion more precise. Once an observer falls inside the black hole she can never come out because to do so she will have to travel faster than the speed of light.

As we have noted already $r = 0$ is a real singularity that is inside the event horizon. Since it is a spacelike surface, once a observer falls insider the event horizon, she is sure to meet the singularity at $r = 0$ sometime in future no matter how much she boosts the rockets.

The summarize, an event horizon is a *stationary, null* surface. For instance, in our example of the Schwarzschild black hole, it is stationary because it is defined as a hypersurface $r = 2GM$ which does not change with time. More precisely, the time-like Killing vector $\frac{\partial}{\partial t}$ leaves it invariant. It is at the same time null because g^{rr} vanishes at $r = 2GM$. This surface that is simultaneously stationary and null, causally separates the inside and the outside of a black hole.

2.5 Black Hole Parameters

From our discussion of the Schwarzschild black hole we are ready to abstract some important general concepts that are useful in describing the physics of more general black holes.

To begin with, a *black hole* is an asymptotically flat spacetime that contains a region which is not in the backward lightcone of future timelike infinity. The boundary of such a region is a stationary null surface call the *event horizon*. The fixed t slice of the event horizon is a two sphere.

There are a number of important parameters of the black hole. We have introduced these in the context of Schwarzschild black holes. For a general black holes their actual values are different but for all black holes, these parameters govern the thermodynamics of black holes.

1. The radius of the event horizon r_H is the radius of the two sphere. For a Schwarzschild black hole, we have $r_H = 2GM$.

2. The area of the event horizon A_H is given by $4\pi r_H^2$. For a Schwarzschild black hole, we have $A_H = 16\pi G^2 M^2$.

3. The surface gravity is the parameter κ that we encountered earlier. As we have seen, for a Schwarzschild black hole, $\kappa = 1/4GM$.

3. Semiclassical Black Holes

3.1 Laws of Black Hole Mechanics

One of the remarkable properties of black holes is that one can derive a set of laws of black hole mechanics which bear a very close resemblance to the laws of thermodynamics. This is quite surprising because *a priori* there is no reason to expect that the spacetime geometry of black holes has anything to do with thermal physics.

(0) Zeroth Law: In thermal physics, the zeroth law states that the temperature T of body at thermal equilibrium is constant throughout the body. Otherwise heat will flow from hot spots to the cold spots. Correspondingly for stationary black holes one can show that surface gravity κ is constant on the event horizon. This is obvious for spherically symmetric horizons but is true also more generally for non-spherical horizons of spinning black holes.

(1) First Law: Energy is conserved, $dE = Tds + \mu dQ + \Omega dJ$, where E is the energy, Q is the charge with chemical potential μ and J is the spin with chemical potential Ω. Correspondingly for black holes, one has $dM = \frac{\kappa}{8\pi G} dA + \mu dQ + \Omega dJ$. For a Schwarzschild black hole we have $\mu = \Omega = 0$ because there is no charge or spin.

(2) Second Law: In a physical process the total entropy S never decreases, $\Delta S \geq 0$. Correspondingly for black holes one can prove the area theorem that the net area in any process never decreases, $\Delta A \geq 0$. For example, two Schwarzschild black holes with masses M_1 and M_2 can coalesce to form a bigger black hole of mass M. This is consistent with the area theorem since the area is proportional to the square of the mass and $(M_1 + M_2)^2 \geq M_1^2 + M_2^2$. The opposite process where a bigger black hole fragments is however disallowed by this law.

Thus the laws of black hole mechanics, crystallized by Bardeen, Carter, Hawking, and other bears a striking resemblance with the three laws of thermodynamics for a body in thermal equilibrium. We summarize these results below in Table 1.

Here A is the area of the horizon, M is the mass of the black hole, and κ is the surface gravity which can be thought of roughly as the acceleration at the horizon[1].

3.2 Hawking temperature

This formal analogy is actually much more than an analogy. Bekenstein and Hawking discovered that there is a deep connection between black hole geometry, thermodynamics and quantum mechanics.

[1]We have stated these laws for black holes without spin and charge but more general form is known.

Table 1: Laws of Black Hole Mechanics

Laws of Thermodynamics	**Laws of Black Hole Mechanics**
Temperature is constant throughout a body at equilibrium. T= constant.	Surface gravity is constant on the event horizon. κ =constant.
Energy is conserved. $dE = TdS$.	Energy is conserved. $dM = \frac{\kappa}{8\pi} dA$.
Entropy never decrease. $\Delta S \geq 0$.	Area never decreases. $\Delta A \geq 0$.

Bekenstein asked a simple-minded but incisive question. If nothing can come out of a black hole, then a black hole will violate the second law of thermodynamics. If we throw a bucket of hot water into a black hole then the net entropy of the world outside would seem to decrease. Do we have to give up the second law of thermodynamics in the presence of black holes?

Note that the energy of the bucket is also lost to the outside world but that does not violate the first law of thermodynamics because the black hole carries mass or equivalently energy. So when the bucket falls in, the mass of the black hole goes up accordingly to conserve energy. This suggests that one can save the second law of thermodynamics if somehow the black hole also has entropy. Following this reasoning and noting the formal analogy between the area of the black hole and entropy discussed in the previous section, Bekenstein proposed that a black hole must have entropy proportional to its area.

This way of saving the second law is however in contradiction with the classical properties of a black hole because if a black hole has energy E and entropy S, then it must also have temperature T given by

$$\frac{1}{T} = \frac{\partial S}{\partial E}.$$

For example, for a Schwarzschild black hole, the area and the entropy scales as $S \sim M^2$. Therefore, one would expect inverse temperature that scales as M

$$\frac{1}{T} = \frac{\partial S}{\partial M} \sim \frac{\partial M^2}{\partial M} \sim M. \tag{3.1}$$

Now, if the black hole has temperature then like any hot body, it must radiate. For a classical black hole, by its very nature, this is impossible. Hawking showed that after including quantum effects, however, it is possible for a black hole to radiate. In a quantum theory, particle-antiparticle are constantly being created and annihilated even in vacuum. Near the horizon, an antiparticle can fall in once in a while and the particle can escapes to infinity. In fact, Hawking's calculation showed that the spectrum emitted by the black hole is precisely thermal with temperature $T = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{8\pi GM}$. With this precise relation between the temperature and surface gravity the laws of black hole mechanics discussed in the earlier section become identical to the laws of thermodynamics. Using the formula for the Hawking temperature and the first law of thermodynamics

$$dM = TdS = \frac{\kappa\hbar}{8\pi G\hbar} dA,$$

one can then deduce the precise relation between entropy and the area of the black hole:

$$S = \frac{Ac^3}{4G\hbar} .$$

3.3 Euclidean Derivation of Hawking Temperature

Before discussing the entropy of a black hole, let us derive the Hawking temperature in a somewhat heuristic way using a Euclidean continuation of the near horizon geometry. In quantum mechanics, for a system with Hamiltonian H, the thermal partition function is

$$Z = \mathrm{Tr} e^{-\beta \hat{H}}, \tag{3.2}$$

where β is the inverse temperature. This is related to the time evolution operator $e^{-itH/\hbar}$ by a Euclidean analytic continuation $t = -i\tau$ if we identify $\tau = \beta\hbar$. Let us consider a single scalar degree of freedom Φ, then one can write the trace as

$$\mathrm{Tr} e^{-\tau \hat{H}/\hbar} = \int d\phi < \phi | e^{-\tau_E \hat{H}/\hbar} | \phi >$$

and use the usual path integral representation for the propagator to find

$$\mathrm{Tr} e^{-\tau \hat{H}/\hbar} = \int d\phi \int D\Phi e^{-S_E[\Phi]}.$$

Here $S_E[\Phi]$ is the Euclidean action over periodic field configurations that satisfy the boundary condition

$$\Phi(\beta\hbar) = \Phi(0) = \phi.$$

This gives the relation between the periodicity in Euclidean time and the inverse temperature,

$$\beta\hbar = \tau \quad \text{or} \quad T = \frac{\hbar}{\tau}. \tag{3.3}$$

Let us now look at the Euclidean Schwarzschild metric by substituting $t = -it_E$. Near the horizon the line element (2.11) looks like

$$ds^2 = \rho^2\kappa^2 dt_E^2 + d\rho^2.$$

If we now write $\kappa t_E = \theta$, then this metric is just the flat two-dimensional Euclidean metric written in polar coordinates provided the angular variable θ has the correct periodicity $0 < \theta < 2\pi$. If the periodicity is different, then the geometry would have a conical singularity at $\rho = 0$. This implies that Euclidean time t_E has periodicity $\tau = \frac{2\pi}{\kappa}$. Note that far away from the black hole at asymptotic infinity the Euclidean metric is flat and goes as $ds^2 = d\tau_E^2 + dr^2$. With periodically identified Euclidean time, $t_E \sim t_E + \tau$, it looks like a cylinder. Near the horizon at $\rho = 0$ it is nonsingular and looks like flat space in polar coordinates for this correct periodicity. The full Euclidean geometry thus looks like a cigar. The tip of the cigar is at $\rho = 0$ and the geometry is asymptotically cylindrical far away from the tip.

Using the relation between Euclidean periodicity and temperature, we then conclude that Hawking temperature of the black hole is

$$T = \frac{\hbar\kappa}{2\pi}. \tag{3.4}$$

3.4 Bekenstein-Hawking Entropy

Even though we have "derived" the temperature and the entropy in the context of Schwarzschild black hole, this beautiful relation between area and entropy is true quite generally essentially because the near horizon geometry is always Rindler-like. For *all* black holes with charge, spin and in number of dimensions, the Hawking temperature and the entropy are given in terms of the surface gravity and horizon area by the formulae

$$T_H = \frac{\hbar\kappa}{2\pi}, \quad S = \frac{A}{4G\hbar}.$$

This is a remarkable relation between the thermodynamic properties of a black hole on one hand and its geometric properties on the other.

The fundamental significance of entropy stems from the fact that even though it is a quantity defined in terms of gross thermodynamic properties it contains nontrivial info about the *microscopic* structure of the theory through Boltzmann relation

$$S = k \log \Omega,$$

where Ω is the total number of microstates of the system of for a given energy, and k is Boltzmann constant. Entropy is not a kinematic quantity like energy or momentum but rather contains information about the total number microscopic degrees of freedom of the system. Because of this relation, can learn a great deal about the microscopic properties of a system from its thermodynamics properties.

The Bekenstein-Hawking entropy behaves in every other respect like the ordinary thermodynamic entropy. It is therefore natural to ask what microstates might account for it. Since the entropy formula is given by this beautiful, general form

$$S = \frac{Ac^3}{4G\hbar},$$

that involves all three fundamental dimensionful constants of nature, it is a valuable piece of information about the degrees of freedom of a quantum theory of gravity.

3.5 Reissner-Nordström Metric

The most general static, spherically symmetric, charged solution of the Einstein-Maxwell theory (2.1) gives the Reissner-Nordström (RN) black hole. In what follows we choose units so that $G = \hbar = 1$. The line element is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \qquad (3.5)$$

and the electromagnetic field strength by

$$F_{tr} = Q/r^2.$$

The parameter Q is the charge of the black hole and M the mass as for the Schwarzschild black hole.

Now, the event horizon for this solution is located at where $g^{rr} = 0$, or

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0.$$

Since this is a quadratic equation in r,

$$r^2 - 2QMr + Q^2 = 0,$$

it has two solutions.

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}.$$

Thus, r_+ defines the outer horizon of the black hole and r_- defines the inner horizon of the black hole. The area of the black hole is $4\pi r_+^2$.

Following the steps similar to what we did for the Schwarzschild black hole, we can analyze the near horizon geometry to find the surface gravity and hence the temperature:

$$T = \frac{\kappa\hbar}{2\pi} = \frac{\sqrt{M^2 - Q^2}}{2\pi(2M(M + \sqrt{M^2 - Q^2}) - Q^2)} \tag{3.6}$$

$$S = \pi r_+^2 = \pi(M + \sqrt{M^2 - Q^2})^2. \tag{3.7}$$

These formulae reduce to those for the Schwarzschild black hole in the limit $Q = 0$.

3.6 Extremal Black Holes

For a physically sensible definition of temperature and entropy in (3.6) the mass must satisfy the bound $M^2 \geq Q^2$. Something special happens when this bound is saturated and $M = |Q|$. In this case $r_+ = r_- = |Q|$ and the two horizons coincide. We choose Q to be positive. The solution (3.5) then takes the form,

$$ds^2 = -(1 - Q/r)^2 dt^2 + \frac{dr^2}{(1 - Q/r)^2} + r^2 d\Omega^2, \tag{3.8}$$

with a horizon at $r = Q$. In this extremal limit (3.6), we see that the temperature of the black hole goes to zero and it stops radiating but nevertheless its entropy has a finite limit given by $S \to \pi Q^2$. When the temperature goes to zero, thermodynamics does not really make sense but we can use this limiting entropy as the definition of the zero temperature entropy.

For extremal black holes it more convenient to use isotropic coordinates in which the line element takes the form

$$ds^2 = H^{-2}(\vec{x})dt^2 + H^2(\vec{x})d\vec{x}^2$$

where $d\vec{x}^2$ is the flat Euclidean line element $\delta_{ij}dx^i dx^j$ and $H(\vec{x})$ is a harmonic function of the flat Laplacian

$$\delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}.$$

The Reissner-Nordström solution is obtained by choosing

$$H(\vec{x}) = \left(1 + \frac{Q}{r}\right),$$

and the field strength is given by $F_{0i} = \partial_i H(\vec{x})$.

One can in fact write a multi-centered Reissner-Nordström solution by choosing a more general harmonic function

$$H = 1 + \sum_{i=1}^{N} \frac{Q_i}{|\vec{x} - \vec{x}i|}. \tag{3.9}$$

The total mass M equals the total charge Q and is given additively

$$Q = \sum Q_i. \tag{3.10}$$

The solution is static because the electrostatic repulsion between different centers balances gravitational attraction among them.

Note that the coordinate r in the isotropic coordinates should not be confused with the coordinate r in the spherical coordinates. In the isotropic coordinates the line-element is

$$ds^2 = -\left(1 + \frac{Q}{r}\right)^2 dt^2 + (1 + \frac{Q}{r})^{-2}(dr^2 + r^2 d\Omega^2),$$

and the horizon occurs at $r = 0$. Contrast this with the metric in the spherical coordinates (3.8) that has the horizon at $r = M$. The near horizon geometry is quite different from that of the Schwarzschild black hole. The line element is

$$\begin{aligned} ds^2 &= -\frac{r^2}{Q^2} dt^2 + \frac{Q^2}{r^2}(dr^2 + r^2 d\Omega^2) \\ &= (-\frac{r^2}{Q^2} dt^2 + \frac{Q^2}{r^2} dr^2) + (Q^2 d\Omega^2). \end{aligned}$$

The geometry thus factorizes as for the Schwarzschild solution. One factor the 2-sphere S^2 of radius Q but the other (r, t) factor is now not Rindler any more but is a two-dimensional Anti-de Sitter or AdS_2. The geodesic radial distance in AdS_2 is $\log r$. As a result the geometry looks like an infinite throat near $r = 0$ and the radius of the mouth of the throat has radius Q.

Extremal RN black holes are interesting because they are stable against Hawking radiation and nevertheless have a large entropy. We now try to see if the entropy can be explained by counting of microstates. In doing so, supersymmetry proves to be a very useful tool.

3.7 Bekenstein-Hawking-Wald Entropy

In our discussion of Bekenstein-Hawking entropy of a black hole, the Hawking temperature could be deduced from surface gravity or alternatively the periodicity of the

Euclidean time in the black hole solution. These are geometric asymptotic properties of the black hole solution. However, to find the entropy we needed to use the first law of black hole mechanics which was derived in the context of Einstein-Hilbert action

$$\frac{1}{16\pi}\int R\sqrt{g}d^4x.$$

Generically in string theory, we expect corrections (both in α' and g_s) to the effective action that has higher derivative terms involving Riemann tensor and other fields.

$$I = \frac{1}{16\pi}\int (R + R^2 + R^4F^4 + \cdots).$$

How do the laws of black hole thermodynamics get modified?

Wald derived the first law of thermodynamics in the presence of higher derivative terms in the action. This generalization implies an elegant formal expression for the entropy S given a general action I including higher derivatives

$$S = 2\pi\int_{\rho^2}\frac{\delta I}{\delta R_{\mu\gamma\alpha\beta}}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\sqrt{h}d^2\Omega,$$

where $\epsilon^{\mu\nu}$ is the binormal to the horizon, h the induced metric on the horizon, and the variation of the action with respect to $R_{\mu\nu\alpha\beta}$ is to be carried out regarding the Riemann tensor as formally independent of the metric $g_{\mu\nu}$.

As an example, let us consider the Schwarzschild solution of the Einstein Hilbert action. In this case, the event horizon is S^2 which has two normal directions along r and t. We can construct an antisymmetric 2-tensor $\epsilon_{\mu\nu}$ along these directions so that $\epsilon_{rt} = \epsilon tr = -1$.

$$\mathcal{L} = \frac{1}{16\pi}R_{\mu\gamma\alpha\beta}g^{\nu\alpha}g^{\mu\beta}, \quad \frac{\partial\mathcal{L}}{\partial R_{\mu\gamma\alpha\beta}} = \frac{1}{16\pi}\frac{1}{2}(g^{\mu\alpha}g^{\gamma\beta} - g^{\nu\alpha}g^{\mu\beta})$$

Then the Wald entropy is given by

$$\begin{aligned} S &= \frac{1}{8}\int\frac{1}{2}(g^{\mu\alpha}g^{\nu\beta} - g^{\nu\alpha}g^{\mu\beta})(\epsilon_{\mu\nu}\epsilon_{\alpha\beta})\sqrt{h}d^2\sigma \\ &= \frac{1}{8}\int g^{tt}g^{rr}\cdot 2 = \frac{1}{4}\int_{S^2}\sqrt{h}d^2\sigma = \frac{A_H}{4}, \end{aligned}$$

giving us the Bekenstein-Hawking formula as expected.

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