QUANTUM PERTURBATIONS - *Lecture 3*

So far, we've discussed just the classical dynamics of the inflaton field ϕ (i.e. its homogeneous solution $\phi(t)$). ϕ governs the energy density of the Early Universe, hence it controls the end of inflation.

$\longrightarrow$ ϕ as a microscopic clock!

Being microscopic ϕ is a quantum object and so it cannot take the same value everywhere. Clock with a certain variance because of Uncertainty principle

$\longrightarrow$ Spatially varying fluctuations $\delta\varphi(\vec{x},t) = \phi(\vec{x},t) - \bar{\phi}(t)$

Because of $\delta\varphi(\vec{x},t)$, different regions of the sky inflate, at the same time, by different amounts.

$\longrightarrow$ Local differences in the time when inflation ends, $\delta t(\vec{x})$.

This generates local changes in the energy density $\rho(\vec{x})$ which are then turned into differences in the CMB temperature, for example.

$\Rightarrow$ In a quantum Universe local fluctuations are unavoidable!

Notice, however, that in doing computations ~~however~~ are less always to think to the inflaton as a clock, not as something that carries energy (We will see it!)

Action for perturbations

What is the action obeyed by $\delta R(\vec{x},t)$?

Assume (for simplicity): gravity is non dynamical, pure dS space

Inflaton action:

$$S[\phi] = \int d^4x\, a^3\left(-\frac{1}{2}\partial_\mu(\bar{\phi}+\delta\phi)\partial_\nu(\bar{\phi}+\delta\phi)g^{\mu\nu} - V(\bar{\phi}+\delta\phi)\right) =$$

$$= S[\bar{\phi}] + S^{(2)}[\delta\phi] + \dots$$

$$S^{(2)}[\delta\phi] = \int d^4x\, \frac{a^3}{2}\left(\dot{\delta\phi}^2 - \frac{(\partial_i\delta\phi)^2}{a^2} - V''(\bar{\phi})\,\delta\phi^2\right)$$

Now $V'' = \frac{3}{2}H^2(\eta - 4\epsilon) \ll 1 \longrightarrow$ slow roll suppressed!

Conclusion: at 1st order in slow roll the quadratic action is the same of a massless scalar field!

Comment:

Metric is dynamical! Perturbations of $\delta\phi$ affect R that in turn changes the action for $\delta\phi$. However, since ~~[illegible]~~ perturbations of the metric probe deviations from dS, they always are slow roll suppressed, so they don't change the leading action!

GR always gives slow-roll suppressed contributions!

Now let us study $S^{(2)}[\delta\phi]$. We use conformal time and we move in Fourier space

- $a = a_0 e^{Ht} \longrightarrow \tau = \int \frac{dt}{a_0 e^{Ht}} = -\frac{e^{-Ht}}{a_0 H} = -\frac{1}{aH}$ $\qquad \tau \in (-\infty, 0)$
- $\delta\phi(\vec{x},t) = \int \frac{d^3x}{(2\pi)^3}\, \delta\phi_{\vec{k}}(\tau)\, e^{i\vec{k}\vec{x}}$

$$S^{(2)}_{[\delta\phi]} = \int d^3x\, dt\, a^3 \left(\dot{\delta\phi}^2 - \frac{(\partial_i \delta\phi)^2}{a^2}\right) = \int d^3x\, d\tau\, a^4 \left(\frac{\delta\phi'^2}{a^2} - \frac{(\partial_i \delta\phi)^2}{a^2}\right) =$$

$$= \int \frac{d^3k}{(2\pi)^3} d\tau\, a^2 \left(\delta\phi_k' \delta\phi_{-k}' + \vec{k}^2 \delta\phi_k \delta\phi_{-k}\right)$$

EoM: $(a^2 \delta\phi_k')' - a^2 \vec{k}^2 \delta\phi_k = 0$

Def: $v_k = a\,\delta\phi_k$

$$\boxed{v_k'' + \left(k^2 - \frac{a''}{a}\right) v_k = 0}$$

Mukhanov-Sasaki eq.

Harmonic oscillator with a time dependent frequency $\omega^2(\tau)$

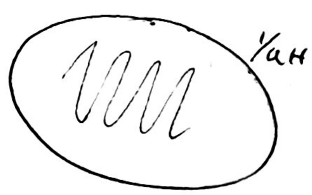
Let's study the equation in the 2 limiting regimes:

$$\left(\frac{a''}{a} = \frac{2}{\tau^2} = (aH)^2 \qquad - \frac{1}{aH}: \text{comoving Hubble horizon}\right)$$

1. $k \gg \frac{a''}{a} \Rightarrow \frac{k^2}{(aH)^2} \gg 1$: Sub-horizon regime

The mode k is deep inside the horizon! $\to$ feels M.

$v_k'' + k^2 v_k = 0 \quad \to \quad v_k(\tau) \propto e^{\pm ik\tau}$



Choose the $(-ik\tau)$ solution and fix the condiz. as in M.

2. $k \ll \frac{a''}{a}$: Super-horizon regime

$$v_k'' - \frac{2}{\tau^2} v_k = 0 \quad \to \quad v_k \propto \begin{cases} 1/\eta & \text{growing mode} \\ \eta^2 & \text{decaying mode} \end{cases}$$

General solution found with Mathematica:

$$v_k^{cl}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right)$$

Recall that $\delta\phi_k = v_k/a$, then:

$$\delta\phi_k^{cl}(\tau) = \frac{H}{\sqrt{2k^3}}(1+ik\tau)e^{-ik\tau}$$

massless scalar field in dS.

<u>Quantization</u>

Canonical quantization: $\delta\phi_{\vec{k}} \to \delta\hat{\phi}_{\vec{k}}$

$$\delta\hat{\phi}_{\vec{k}} = \delta\phi_k^{cl}(\tau)\,\hat{a}^+_{\vec{k}} + \left(\delta\phi_k^{cl}(\tau)\right)^*\hat{a}_{-\vec{k}}$$

The coefficients of the Fourier expansion have been promoted to the creation and annihilation operators of the harmonic oscillator.

$$\to [\hat{a}_{\vec{k}}, \hat{a}^+_{-\vec{p}}] = \delta(\vec{k}-\vec{p}), \qquad [\hat{a}^+_{\vec{k}}, \hat{a}^+_{\vec{p}}] = [\hat{a}_{\vec{k}}, \hat{a}_{\vec{p}}] = 0$$

Define the vacuum state demanding $\hat{a}_{\vec{k}}|0\rangle = 0$ and then construct the Hilbert space acting with $\hat{a}^+_{\vec{k}}$:

$$|m_{\vec{k}_1}, n_{\vec{k}_2}, \dots\rangle = \frac{1}{\sqrt{m!\,n!\cdots}}(\hat{a}^+_{\vec{k}_1})^m(\hat{a}^+_{\vec{k}_2})^n\cdots|0\rangle$$

After having quantized the system we can compute the 2-point function of the field. (expectation value of $\delta\phi^2$):

$$\langle\delta\phi_{\vec{k}}\delta\phi_{\vec{p}}\rangle = \langle 0|\left(\hat{a}_{\vec{k}}\delta\phi_k + \hat{a}^+_{-\vec{k}}\delta\phi^*\right)\left(\hat{a}_{\vec{p}}\delta\phi_p + \hat{a}^+_{\vec{p}}\delta\phi^*\right)|0\rangle =$$

$$= \delta\phi_k\,\delta\phi_p^*\,\langle 0|\hat{a}_{\vec{k}}\hat{a}^+_{\vec{p}}|0\rangle =$$

$$= (2\pi)^3\delta(\vec{k}-\vec{p})\,|\delta\phi_k^{cl}|^2$$

Observables are related to the $\langle\delta\phi_k\delta\phi_p\rangle$ 2-point function evaluated at the end of inflation, i.e. at $\tau \to 0$

Def: power spectrum $P_{\delta\phi}(k) = |\delta\phi_k^{ce}(0)|^2 = \frac{H_*^2}{2k^3}$

$H_* \equiv H(\tau_*)$ / $k\tau_* = 1$

Connection with late time observables (CMB T fluctuations)

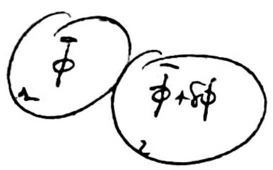
Because of the quantum fluctuations $\delta\phi$, in different regions of the sky inflation ends at different times.

$\Rightarrow$ At a fixed time different regions expand in different way!

$\rightarrow a(t) \rightarrow a(t, \vec{x})$

How can we quantify the differences in the expansion rate?

Consider 2 different patches:



I can see the region 1 as slightly advanced/retarded with respect to 2.

$$\phi(t) + \delta\phi(t, \vec{x}) = \phi(t - \delta t)$$

$$\Rightarrow \boxed{\delta t(t, \vec{x}) = -\frac{\delta\phi(t, \vec{x})}{\dot{\phi}}}$$

Then the spatial part of the metric is:

$$g_{ij} = a(t)^2 \delta_{ij} \longrightarrow g_{ij} = a(t + \delta t)^2 \delta_{ij} = $$

$$\overset{!}{=} a(t)^2 \big(1 \underbrace{- 2\frac{H}{\dot{\phi}}\delta\phi}_{2\zeta(t,\vec{x})}\big) \delta_{ij}$$

Def: $\zeta(t, \vec{x}) \equiv -\frac{H}{\dot{\phi}} \delta\phi(t, \vec{x})$

Def: power spectrum $P_{\delta\phi}(k) = |\delta\phi_k^{cl}(0)|^2 = \frac{H_*^2}{2k^3}$

H_*: value of the Hubble rate when the mode freeze $(k\tau = 1)$.

<u>Connection with late time observables</u> (<u>CMB temp. fluctuations</u>)
equivalent to

The calculation we carried out so far has been done in a gauge in which the spatial part of the metric is unperturbed.

$$g_{ij} = a^2 \delta_{ij} \qquad \phi = \bar{\phi} + \delta\phi$$

It turns out that to describe CMB fluctuations we need to go in ~~the so called~~ a gauge in which the inflaton is constant on spatial hypersurfaces.

$$\phi = \text{const hypersurface} \longrightarrow \text{constant } T \text{ surfaces in RD.}$$

The 2 gauges are related by a time diff: $\delta t = -\frac{\delta\phi(\vec{x},t)}{\dot{\phi}}$. This changes the spatial part of the metric:

$$g_{ij} = a^2(t+\delta t)\,\delta_{ij} = a^2\left(1 - 2\frac{H\,\delta\phi}{\dot{\phi}}\right)\delta_{ij} \qquad \phi = \bar{\phi}$$

Def: $\zeta(\vec{x},t) = -\frac{H}{\dot{\phi}}\,\delta\phi(\vec{x},t)$ curvature perturbations of constant inflaton hypersurfaces $\left({}^3R = -\frac{4}{a^2}\vec{\nabla}^2\zeta\right)$

$$\boxed{\zeta \longrightarrow \frac{\Delta T}{T} \text{ in RD}}$$

<u>Comment</u>: $\delta\phi_k$ are time indep outside the horizon only at leading order in slow roll. The slow roll suppressed ones make them decay outside the horizon. ζ_k can be shown that are time indep at any order in slow roll, so they do not depend on the <u>reheating</u> details!

Power spectrum of ζ:

$$P_\zeta = \frac{H_*^2}{\dot\phi^2} P_{\delta\phi}(k) = \frac{H_*^2}{4\epsilon M_{pl}^2} \cdot \frac{1}{k^3}$$

$P_\zeta \propto 1/k^3$ → scale invariant as the power spectrum of $\frac{\Delta T}{T}$.

Small deviation from scale invariance: H_* is not perfectly constant!

<u>Deviation from scale invariance:</u>

Def: dimensionless power spectrum: $\Delta_\zeta^2 = \frac{k^3}{2\pi^2} P_\zeta(k) = \frac{H_*^2}{8\pi^2 \epsilon M_{pl}^2}$

scalar tilt: $(n_s - 1) := \frac{d\log \Delta_\zeta^2}{d\log k}$

H_* is the value of H when k leaves the horizon, i.e. $|k\tau| = 1$

$|k\tau| = 1 \Rightarrow k = aH$

differentiate

$$d\ln k = d\ln(aH) = \frac{1}{aH}\left(H\frac{da}{dt}dt + \frac{dH}{dt}dt\right) = H dt(1-\epsilon)$$

$$\Rightarrow n_s - 1 = \frac{d\log\Delta_\zeta^2}{H dt} = \frac{1}{H}\cdot\frac{\epsilon}{H^2}\cdot\left(\frac{H^2}{\epsilon}\right)^{\cdot} = \frac{1}{H}\cdot\frac{\epsilon}{H^2}\left(2\frac{H\dot H}{\epsilon} - \frac{H^2\dot\epsilon}{\epsilon^2}\right) = -2\epsilon - \eta$$

$\boxed{n_s - 1 = -2\epsilon - \eta}$ → the scalar tilt is slow roll suppressed!

Experimentally: $n_s - 1 = 0.032 \pm 0.006$ (Planck 2015)