Introduction to Cosmology

Petnica Summer School 2019

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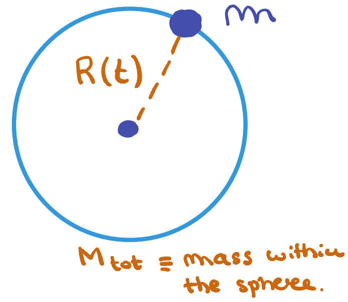
Few references:

- D. Baumann, lecture notes on Cosmology
- V. Mukhanov, “Physical Foundations of Cosmology”
- S. Weinberg, “Cosmology”
- S. Dodelson, “Modern Cosmology”

Friedmann Eq. From Newtonian Gravity:

Consider a Universe filled by non-relativistic ($v \ll c$) particles of mass m.
Assume they are distributed ISOTROPICALLY and HOMOGENEOUSLY.
This means we can define an ENERGY or MASS DENSITY $\rho(t)$.

We can pick an arbitrary sphere of radius $R(t) = a(t)\,\chi_{in}$
and ask what happens to a particle positioned at that distance.
The particle feels the Newtonian attraction of the particles inside the sphere. The net contribution of the particles outside is zero (justified by Birkoff's theorem in GR).



$$m\ddot{R}(t) = -\frac{G\,m\,M_{tot}}{R^2(t)} = -\frac{4\pi G}{3}\, m\, \underbrace{\boxed{\frac{M_{tot}}{\frac{4\pi}{3}R^3(t)}}}_{\text{mass-density } \rho(t)} R(t)$$

1) $\cancel{m}\,\ddot{a}(t)\,\cancel{\chi_{in}} = -\frac{4\pi G}{3}\,\cancel{m}\,\rho(t)\,a(t)\,\cancel{\chi_{in}}$

$$\Downarrow$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho(t)}$$ Friedmann Equation ①
(modified in GR)

Go back to (1): by multiplying by $\dot{a}(t)$ we can get an equation for the energy

$$\rho = \frac{M_{tot}}{\frac{4\pi}{3}R^3(t)} = \frac{M_{tot}}{\frac{4\pi}{3}R(0)^3}\,\frac{R^3(0)}{R^3(t)} = \rho_0 \cdot \left(\frac{a_0}{a(t)}\right)^3$$

$$\dot{a}\ddot{a} = \frac{d}{dt}\frac{\dot{a}^2}{2} = -\frac{4\pi G}{3}\rho\, a\dot{a} = -\frac{4\pi G}{3}\rho_0 \underbrace{\left(\frac{a_0}{a}\right)^3 a\dot{a}}_{= a_0^3 \frac{\dot{a}}{a^{2}} = a_0^3 \frac{d}{dt}\left(-\frac{1}{a}\right)} = \frac{4\pi G}{3}\rho_0 a_0^3 \frac{d}{dt}\left(a^{-1}\right)$$

- Integrate both sides:

$$\frac{\dot{a}^2}{2} = \frac{4\pi G}{3}\rho_0 a_0^3 \frac{1}{a} + U$$

$$\frac{\dot{a}^2}{2} + V(a) = U \quad \text{Energy conservation}$$

$$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho(t) + \frac{2U}{a^2}}$$ Friedmann Equation ②
(True in GR)

- Similar to rocket launched from Earth: the rocket flies away if the initial velocity is above a critical velocity V_{cr}.
- For the sphere to recollapse we need $\dot{a} = 0$ at some point: we need $U < 0$.
  If $U > 0$ the sphere expands forever ($\dot{a} \neq 0$).
- Hubble function: $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$ rate of expansion of the universe.
- We can find the critical density ρ_{cr} by setting $U = 0$

$$\rho_{cr} = \frac{3H^2}{8\pi G};$$

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{cr}(t)} \quad \begin{cases} > 1 \Rightarrow U < 0 & \text{(recollapse)} \\ = 1 \Rightarrow U = 0 & (\dot{a} \to 0,\ t \to \infty) \\ < 1 \Rightarrow U > 0 & \text{(expansion)} \end{cases}$$

GR Effects:

[UNITS: $1\,eV = 1.2\cdot 10^{-6}$ meters $= 4\cdot 10^{-15}$ seconds]

a) In SR/GR, $\rho(t)$ is not the mass-density but the ENERGY DENSITY

$$E = \sqrt{m^2c^4 + \vec{p}^{\,2}c^2}$$

(set $c=1$ from now)

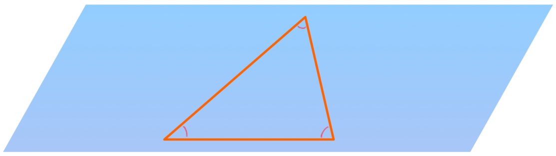
PHOTONS: $m=0$, $E\neq 0$ contribute to expension of the Universe!

We can have many species: MATTER, RADIATION, .. contributing to to the expension

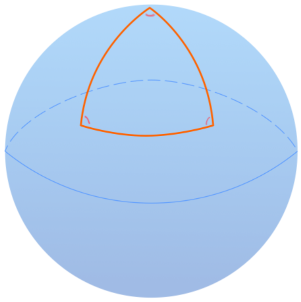
b) In GR the constant of integration U is replaced by the SPATIAL CURVATURE $K = -2U$

K has a geometrical meaning: like the curvature of a 2-sphere but here in 3-d.

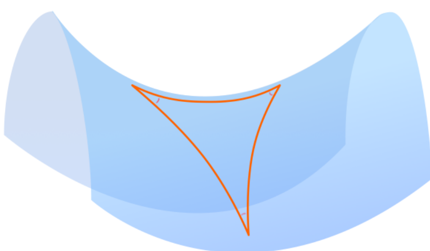
$K=0$ flat space



$K>0$ 3-sphere (positively curved)



$K<0$ hyperbolic / open (negatively curved)



- Curvature K can be probed looking at Fried. equation

More importantly, can be measured by direct geometrical effects.

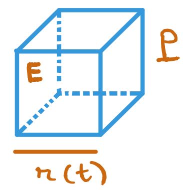
Experimentally very small (close to zero).

Affects Fried. Eq^n with less than 1%.

• Continuity equation:

We also need an equation for $\rho(t)$ so that we can solve the Fried. eq$^{\underline{n}}$.

In a volume V, in absence of pressure, $E = \rho \cdot V$ is conserved



If $P \neq 0$ the 1st law of thermodynamics gives

$$dE = -P\,dV$$

For expanding universe the variation is w.r.t. time t.

$$\dot{E} = -P\,\dot{V}$$

$$\rho = \frac{E}{V} \Rightarrow \dot{\rho} = \left(\frac{E}{V}\right)^{\bullet} = \frac{\dot{E}}{V} - \frac{E}{V}\frac{\dot{V}}{V} = \frac{-P\dot{V}}{V} - \rho\frac{\dot{V}}{V} = -\frac{\dot{V}}{V}(P+\rho)$$

If universe expands $r(t) = a(t)\,r_0 \Rightarrow V(t) = V_0\,a(t)^3$; $\frac{\dot{V}}{V} = 3\frac{\dot{a}}{a} = 3H$

$$\boxed{\dot{\rho} = -3H(\rho + P)}$$

EXAMPLES:

a) **MATTER:** $P \approx 0$; $\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \Rightarrow \rho = \rho_0\left(\frac{a_0}{a}\right)^3$

b) **RADIATION FLUID:** $P = \frac{1}{3}\rho$; $\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a} \Rightarrow \rho = \rho_0\left(\frac{a_0}{a}\right)^4$

Recall that:

$$P = N \cdot \frac{\text{Force}}{\text{Area}} = N\frac{\Delta p}{\Delta t}\frac{1}{\text{area}} = \frac{N}{A}\frac{\Delta p_x}{\Delta L_x/\langle v_x\rangle} = \frac{N}{A}\frac{\Delta p_x}{\Delta L_x}\frac{1}{\sqrt{3}} = \frac{N}{\sqrt{3}}\frac{\Delta p_x\langle v\rangle}{\text{vol}}$$

$$\langle v^2\rangle = 3\langle v_x^2\rangle = 1$$
$$\langle \Delta p^2\rangle = 3\langle \Delta p_x^2\rangle = 1$$

$$= \frac{N}{3}\frac{\langle \Delta p\, v\rangle}{\text{vol}} = \frac{1}{3}\rho \quad \blacksquare$$

Equation Recap:

i) $\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho(t) - \frac{K}{a^2}}$ **FRIEDMANN EQ.**

ii) $\boxed{\dot{\rho} = -3H(\rho + P)}$ **CONTINUITY EQ.**

iii) $\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)}$ **"ACCELERATION" EQ.:** Consistency with (i),(ii) implies we need a term $\propto P(t)$.

EXERCISE: Obtain EQ.(iii) from (i) and (ii).

$$\frac{d(i)}{dt} \to \dot{H^2} = 2H\dot{H} = 2\frac{\dot{a}}{a}\left(\frac{\ddot{a}a - \dot{a}^2}{a^2}\right) = \frac{8\pi G}{3}\dot{\rho} + \frac{2K}{a^2}H$$

$$= 2H\left(\frac{\ddot{a}}{a} - H^2\right) = \frac{8\pi G}{3}\dot{\rho} + \frac{2KH}{a^2}$$

$$\overset{(i)}{=} 2H\left(\frac{\ddot{a}}{a} - \frac{8\pi G}{3}\rho + \cancel{\frac{K}{a^2}}\right) \overset{(ii)}{=} \frac{8\pi G}{3}\left(-3H(\rho+P)\right) + \cancel{\frac{2KH}{a^2}}$$

$$\Rightarrow \cancel{2}\cancel{H}\frac{\ddot{a}}{a} = -\overset{4}{\cancel{8}}\pi G\,\cancel{H}(\rho+P) + \cancel{2}\cancel{H}\,\frac{8\pi G}{3}\rho$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(-2\rho + 3\rho + 3P) = -\frac{4\pi G}{3}(\rho + 3P) \quad \blacksquare$$

• COMMENTS about (iii): It describes the acceleration / deceleration of the Universe. Normal substances give deceleration ($\rho + 3P > 0$).

This is intuitive: deceleration $\longleftrightarrow$ attractive force of gravity.

• NON-RELATIVISTIC IDEAL GAS: $PV = k_B N T$

avg. kin energy ~ mv^2

$$P = k_B \frac{N}{V} T = \frac{\rho}{m} k_B \cdot T \sim \rho \frac{(k_B \cdot T)}{m} \sim \rho \frac{m\langle v^2 \rangle}{m} \ll \rho$$

$\Rightarrow$ For non-rel. systems the pressure is negligible.

$\left(\frac{\langle v^2 \rangle}{c^2} \longrightarrow \infty \right)$

Hubble Law:

Usually, the HUBBLE LAW is phrased as : $v(t) = \frac{d}{dt}(r(t)) = \frac{d}{dt}(a(t)r_c) = \frac{\dot{a}}{a} a r_c = H_0 r$

Far-away object are seen moving faster than close-by ones.
However, velocities are difficult to be measured directly. What is measured is the **REDSHIFT** of emission/absorption lines. The redshift (called z) is analogous to the velocity.
The distance instead can be obtained by knowing the LUMINOSITY of an object.

$$z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} ,$$

For non-rel. motion $z = v/c$.
The Hubble-law can be phrased as

$$z = \frac{H_0 r}{c} ,$$ works for close-by objects !

Indeed the wave-lenght depends on the scale factor $\lambda = a(t)\lambda_c$

$$1+z = \frac{\lambda_{obs}}{\lambda_{em}} = \frac{a_{obs}}{a_{em}}$$

For far objects light travels for a relevant portion of the hystory of the Universe.
Since $H(t) \neq H_0$ the redshift becomes more complicated.

The H_0 today can be obtained by fitting to this relation.

$$H_0 \simeq 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$$ [1pc = $3.086 \cdot 10^{16}$ meters]

Luminosity Distance:

Consider an object of LUMINOSITY L at some distance d in expanding universe.

$$L = \frac{\Delta E_{em}}{\Delta t_{em}}$$ (energy emitted per unit-time at the emission).

In non-expanding space the observed flux F for an object of luminosity L at a distance d is $F = \frac{L}{4\pi d^2}$.

Similarly, for expanding universe we define a "LUMINOSITY DISTANCE" d_L as it would be in flat space:

$$d_L \equiv \left(\frac{L}{4\pi F}\right)^{\frac{1}{2}}$$, This ratio is measurable.

The distance evolves as the photons propagate, so can be defined only for close objects.

$d_L \neq R$

$$d_L = (1+z) R = (1+z) a_0 R_c$$

This result can be understood by looking at the definition of d_L:
- one $(1+z)$ enters because of the redshift of photons $\omega \propto 1/a$.
- one $(1+z)$ enters since the distance increases with a.

These two effects make the flux "lower", so that objects appear dimmer.
d_L can be used to probe the evolution of $a(t)$:
For flat universe we can relate R with z (and so d_L with z).

Side note: physical distances are calculated with a line-element / metric
In relativity : $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$
In FLRW we have a scale factor : $ds^2 = -dt^2 + a(t)^2 \left[dx^2 + dy^2 + dz^2 \right]$

Photons travel along paths with $ds^2 = 0$ (speed of light).
$dt = dx$ (SR) $\longrightarrow$ $dt = a(t)\,dx$ (FLRW)

$$\Delta R_c = \frac{\Delta t}{a(t)} \Rightarrow R_c = \int_{t_{em}}^{t_{obs}} \frac{dt}{a(t)} = \int_{a_{em}}^{a_{obs}} \frac{1}{a}\frac{dt}{da}\,da = \int_{a_{em}}^{a_{obs}} \frac{1}{a}\frac{1}{\dot a}\,da = \int_{a_{em}}^{a_{obs}} \frac{da}{a^2 H} \quad ; \quad \frac{a}{a_0} = \frac{1}{1+z}$$

$$da/a_0 = \frac{-1}{(1+z)^2}\,dz$$

$z_{obs} = 0$
$z_{em} = z$

$$= \int_{a_{em}}^{a_{obs}} \frac{da/a_0}{a^2/a_0^2\, H}\,\frac{a_0}{a_0^2} = -\frac{1}{a_0}\int_z^0 \frac{dz'\,(1+z')^2}{(1+z')^2\,H(z')}$$

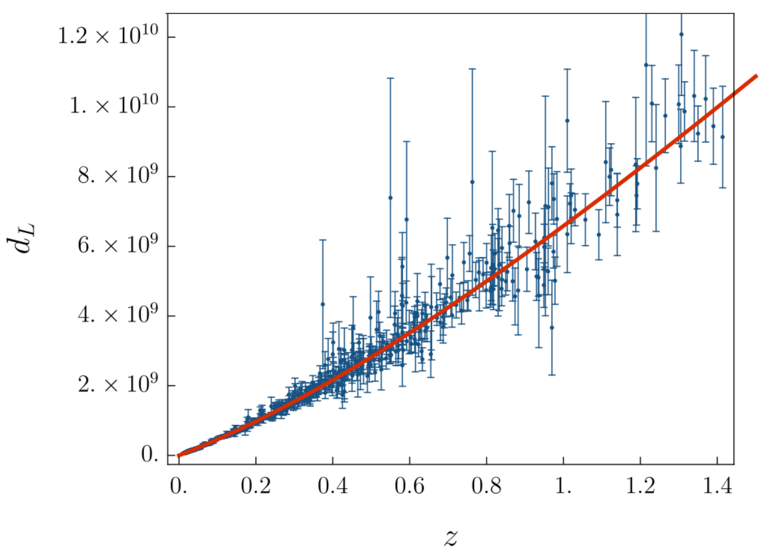
$$= \frac{1}{a_0}\int_0^z \frac{dz'}{H(z')}$$

- $R = a_0 R_c = \displaystyle\int_0^z \frac{dz'}{H(z')}$

- $d_L = 1+z \displaystyle\int_0^z \frac{dz'}{H(z')}$, the integral makes d_L sensitive to the whole hystory of the Universe.

EXERCISE : $H(z) = H_0 \left[\Omega_m (1+z)^3 + \Omega_\Lambda \right]^{\frac{1}{2}}$ (*) $H(z)$ as a function of redshift for MATTER + COSMOLOGICAL CONSTANT (no radiation or curvature).

With experiments we can obtain $d_L(z)$ and then fit (*) to obtain the values for $\Omega_M, \Omega_\Lambda, H_0$ (best fit).
See Mathematica Notebook / Tanja exercises



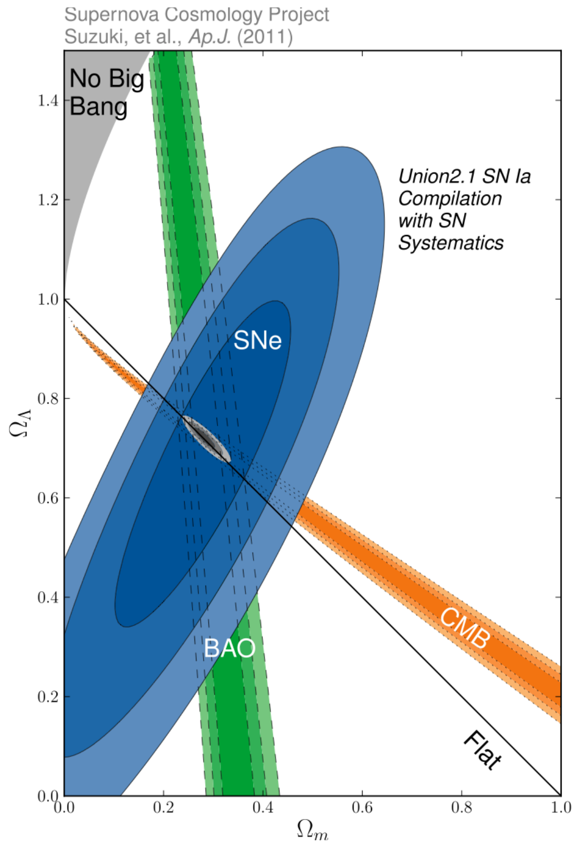
NB: data are given in magnitude vs z.

The magnitude is related to d_L through the relation

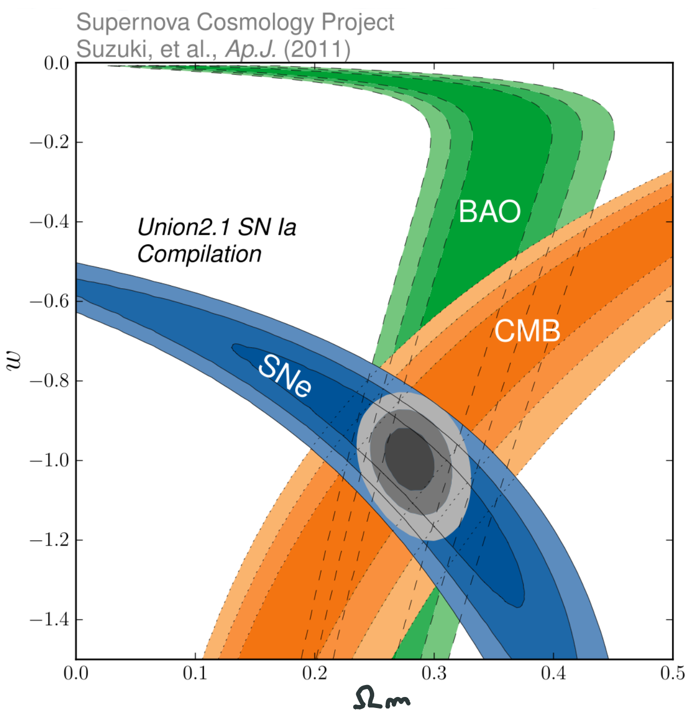
$$d_L \equiv 10\,\mathrm{pc} \cdot 10^{\frac{m}{5}}$$

$H_0 \simeq 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$

(data come from Supernovae type Ia).
SNe Ia



Stronger evidence for the cosmological constant comes from CMB.
By combining CMB & SNe Ia we get tight constraints on the values of Ω_Λ & Ω_m.
Indeed CMB is mainly sensitive to curvature so the constrain is along the "flat universe" line $\Omega_m + \Omega_\Lambda = 1$.
On the other hand SNe have a different dependence that helps to break the degeneracies between Ω_m & Ω_Λ.



SNe Ia and CMB give also good evidence for an equation of state for DARK ENERGY
$\omega \simeq -1$.
Notice that to explain acceleration one needs only $\omega < -\frac{1}{3}$.

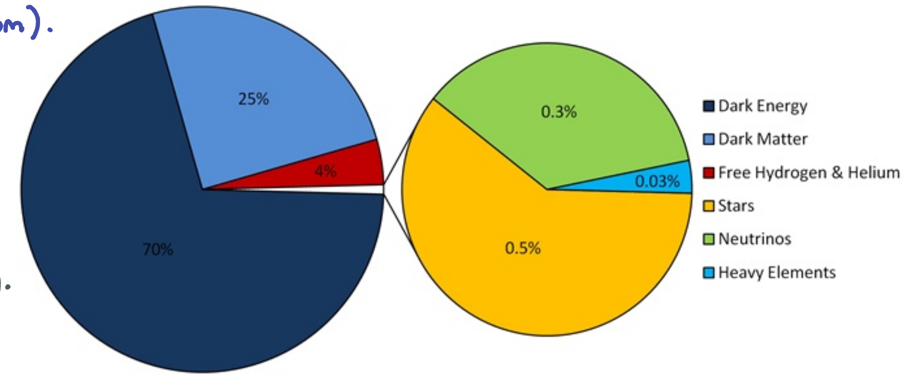
These data are really telling us we are dealing with a cosmological constant, not with some other exotic form of matter.

COMPOSITION OF THE UNIVERSE

Most of the conclusions we can draw about the behaviour / dynamics of particles in the early universe are based on well known physics (which we believe is under control). For example, as we will see, CMB physics is really atomic physics. What we can probe directly extends back to energies of ~MeV scales (nuclear physics).

Below these energies, the main (energy) components of the UNIVERSE are PHOTONS, BARYONS, ELECTRONS, NEUTRINOS, DARK MATTER & DARK ENERGY.

• PHOTONS (primordial radiation).
Make the CMB, with a temperature today $T_{CMB} \simeq 2.7K$. Today, this corresponds with around 10^{-5} of the total energy.
This radiation has a perfect Planckian spectrum (black body).



• BARYONS: In cosmology baryons are basically everything we can "see" directly. In other words planets, stars, clouds are made of baryons. Essentially they are just neutral hydrogen.
$\Omega_b^0 = 0.04$.
There are much more photons than baryons.

• DARK MATTER: From observations we have that curvature is negligible : this implies we need to "add" forms of energy to make $\sum_i \Omega_i^0 = 1$.
Most of the energy budget of the universe is thus in DARK COMPONENTS.
Dark Matter is one of these components. Gravitationally it behaves as regular baryonic matter (pressureless, cold, dust)
However, since we don't see it and don't detect it, it must be weakly coupled with normal matter (standard model particles).
Overall $\Omega_{DM}^0 \simeq 0.25$.
Dark matter is essential for the formation of structures in the universe: each galaxy is supposedly made out of a DM "halo" surrounding the usual "disk" of stars. (DM can cluster).
POSSIBLE EXPLANATIONS FOR DM: WIMP (weakly interacting massive particles), SUSY particles (supersymmetric), AXIONS, Primordial Black Holes...
Still we don't know.

• DARK ENERGY: Most abundant energy today $\Omega_{DE}^0 \simeq 0.7$. It can be a cosmological constant ($w_{DE} = -1$) or an even more exotic form of energy. It does not cluster (contrarily to DM).
We have no idea why DE dominates exactly now and why it takes the values it has...

• RELIC NEUTRINOS: they are relics of the "BIG BANG" (like photons). We know they are massive (very light) and they constitute $\lesssim 1\%$ of energy today.

- A cosmological constant is necessary to fit data : $\Omega_\Lambda \simeq 0.7$
  - Peculiar form of energy with equation of state
    $\rho_\Lambda = -P_\Lambda$, $w = -1$

In Fried. eq^{ns} it leads to acceleration :

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_\Lambda + 3P_\Lambda) = -\frac{4\pi G}{3}(-2)\rho_\Lambda = \frac{8\pi G}{3}\rho_\Lambda > 0$$

$$\dot{\rho}_\Lambda = -3H(\rho_\Lambda + P_\Lambda) = 0 \quad \Rightarrow \quad \rho_\Lambda = \text{const.}$$

· Other forms of energy decay in time : Λ dominates at late times in the Universe (from now on)

- Λ is compatible with the principles of GR (actually introduced by EINSTEIN, hoping to make the universe static).

- The theoretical estimate for its value is off by several (60) orders of magnitude !

EXAMPLE : The contributions from the STANDARD MODEL are way larger than what we measure. For instance the VACUUM ENERGY of the HIGGS is (expected to be) around

$$V(h) \sim (m_h)^4 \sim (125\,\mathrm{GeV})^4$$

- $$\frac{\rho_\Lambda}{V(h)} \simeq \frac{3H_0^2}{8\pi G}\frac{1}{(m_h)^4} \simeq \frac{H_0^2 M_P^2}{(m_h)^4} \simeq \left(\frac{10^{-2}\,10^{-9}\,\mathrm{GeV}}{125\,\mathrm{GeV}}\right)^4 \sim 10^{-36}$$

[LHC probes physics $\sim 10\,\mathrm{TeV}$ so we actually expect larger contributions (sum over zero point energies).]

It seems there is a "conspiracy" for which these contributions cancel between each other.

Many attempt to solve this conceptual issue, no good answer so far...

- One possible "explanation" is the ANTHROPIC PRINCIPLE : if Λ was larger the universe would be much different and structures (galaxies & stars) wouldn't have formed. The only universes compatible with what we see (stars & planets) are those with a small Λ.

By accepting this, however, we don't have additional insights on its nature... still an open problem.

Equilibrium Thermodynamics

[briefly done during lectures].

From observations (mainly CMB, as we will see) there is good evidence the universe, at early stages, was at thermal equilibrium (LOCAL THERMAL EQUILIBRIUM). Densities of species follow thermodynamical distributions.

- Macroscopic quantities can be computed from statistical distributions.

For a gas, the state is described by positions and momenta of all particles. Statistically, we can describe the system according to the distribution function $f(\underbrace{\vec{x},\vec{p}}_{\text{phase - space variables}},\underbrace{t}_{\text{the distribution can evolve in time}})$ of these particles.

Because of HOMOGENEITY & ISOTROPY we have respectively:
- f is independent on $\vec{x}$
- f depends only on $|\vec{p}| \equiv p$.

In quantum mechanics momentum eigenstates are discrete. This fixes the density of states (think of a particle in a box) as $\frac{1}{(2\pi\hbar)^3}\cdot g$

number of internal degrees of freedom (spin)

With these ingredients we can write the NUMBER DENSITY (particles per unit volume) as an integral over momenta:

$$n \equiv \frac{g}{(2\pi\hbar)^3}\int d^3p\, f(t,p)$$

The energy density ρ can be obtained by integrating over $f\cdot E(p)$:

$$\rho \equiv \frac{g}{(2\pi\hbar)^3}\int d^3p\, f(t,p)\,E(p) \quad ; \quad \text{for weakly interacting particles } E(p)=\sqrt{m^2+p^2}$$

Finally, the pressure P takes the form

$$P \equiv \frac{g}{(2\pi\hbar)^3}\int d^3p\, f(t,p)\,\underbrace{\frac{p}{E(p)}}_{\text{velocity}}\cdot\underbrace{\frac{p}{3}}_{\text{not all particles at fixed v reach the surface area.}}$$

- We can set $\hbar = 1$. the integral can be simplified to $\int d^3p = 4\pi\int_0^\infty dp$.

EQUILIBRIUM DISTRIBUTIONS:

If the collisions between particles are frequent enough, the we quickly reach (local) thermal equilibrium. The thermal distribution depends on the statistics of the species we consider (FERMI - DIRAC or BOSE - EINSTEIN):

$$f(p) = \frac{1}{e^{(E(p)-\mu)/k\cdot T} \pm 1} \quad ; \quad + : \text{fermion} \qquad - : \text{boson}$$

We can also choose units where $k_B = 1$.
At low temperatures we recover Maxwell-Boltzmann distribution.

The coefficient $\mu \equiv$ CHEMICAL POTENTIAL, can be T-dependent. It enters in the 2nd law of thermodynamics:

$$T dS = dU + P dV - \mu \underset{\text{number of particles}}{dN} \quad ; \quad \mu = -T \left(\frac{\partial S}{\partial N}\right)\Big|_{U,V}$$

Let's see how n, ρ, P behave in the relativistic & non-relativistic limits:

· high temperature limit: (RELATIVISTIC) $E(p) = \sqrt{m^2 + p^2} \approx p \quad ; m \ll T ; \mu = 0$

- $$\rho = \frac{g}{(2\pi)^3} \int d^3p \, \frac{E(p)}{e^{E/T} \pm 1} \approx \frac{g}{(2\pi)^3} 4\pi \int_0^\infty p^3 \frac{1}{e^{p/T} \pm 1} dp = \frac{g}{2\pi^2} T^4 \int_0^\infty dz \, \frac{z^3}{e^z \pm 1} \quad ; \quad \frac{p}{T} \equiv z$$

$$= \frac{\pi^2}{30} g T^4 \cdot \begin{cases} 1 & \text{BOSONS} \\ \frac{7}{8} & \text{FERMIONS} \end{cases}$$

(Stefan-Boltzmann: for photons (bosons) we get $g = 2$).

- $$n = \frac{g}{(2\pi)^3} \int d^3p \, \frac{1}{e^{E/T} \pm 1} \approx \frac{g}{(2\pi)^3} 4\pi \int_0^\infty p^2 \frac{1}{e^{p/T} \pm 1} dp = \frac{\zeta(3)}{\pi^2} g T^3 \cdot \begin{cases} 1 & \text{BOSONS} \\ \frac{3}{4} & \text{FERMIONS} \end{cases}$$

$\zeta(3) \simeq 1.202$

one can show $P = \frac{\rho}{3}$; $w = \frac{1}{3}$ (radiation fluid)

· low temperature limit: (NON-RELATIVISTIC), $E(p) = \sqrt{m^2 + p^2} \approx m + \frac{p^2}{2m} \quad ; m \gg T ; \mu = 0$

- $$n \approx \frac{g}{2\pi^2} \int_0^\infty p^2 \frac{1}{e^{(m + p^2/2m)/T}} dp = \frac{g T^3}{2\pi^2} e^{-x} \int_0^\infty z^2 e^{-z^2/(2x)} dz \quad ; \quad x \equiv \frac{m}{T} \; ; \; z \equiv \frac{p}{T}$$

$$= \frac{g T^3}{2\pi^2} e^{-x} \cdot \sqrt{\frac{\pi}{2}} \, x^{3/2} = g \left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} e^{-m/T}$$

At low temperatures massive particles $(m \gg T)$ are exponentially rare, as is expected.
$E \approx m \Rightarrow \rho \simeq m \cdot n$

- $P \simeq n \cdot T$ (* exercise: convince yourself this is correct)

$$w = \frac{P}{\rho} \simeq \frac{T}{m} \ll 1 \Rightarrow \text{matter fluid: pressureless fluid (dust).}$$

· Difference between fermions & bosons important at high temperatures, but irrelevant in non-relativistic limit (at the level of distributions).

Entropy Conservation: (optional)

The 2^{nd} law of thermodynamics tells us that the entropy always increases (or stays constant). At equilibrium the entropy stays constant and we can exploit this fact to track the evolution of certain species.

Recall the continuity equation $\dot{\rho} + 3H(\rho + P) = 0$; $U = \rho V$

$$\frac{dS}{dt} = \frac{1}{T}\left[\frac{dU}{dt} + P\frac{dV}{dt}\right] = \frac{1}{T}\left[\dot{\rho}V + \rho\dot{V} + P\dot{V}\right] = \frac{V}{T}\left[\dot{\rho} + \frac{\dot{V}}{V}(\rho + P)\right]$$

$$V = a^3 V_0 \; ; \; \frac{\dot{V}}{V} = 3H \qquad = \frac{V}{T}\left[\dot{\rho} + 3H(\rho + P)\right] = 0$$

We can define an **ENTROPY DENSITY** $s \equiv \frac{S}{V}$.

It obeys $\dot{s} + 3Hs = 0 \Rightarrow s \propto a^{-3}$ (dilution by volume only)

Radiation Era:

In the early stages the universe is dominated by radiation. As we have seen the energy density for radiation scales as $\rho \sim a^{-4}$.
As we will see more in details at this point species are in thermal equilibrium: the rate of interactions is large enough and species follow their thermal distribution.

For instance, if $\rho \propto a^{-4}$ and $\rho \propto T^4$ then $T = T(t)$ and the temperature can be used as a "clock".

$T(t) = T_0 \frac{a_0}{a(t)} = T_0(1+z)$ (where t_0 is some reference time).
Temperature redshifts like energy.

$$H^2 = \frac{8\pi G}{3}\rho_{rad} = \frac{8\pi G}{3}\Omega^0_{rad}\frac{a_0^4}{a^4} \; ; \; H = \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\Omega^0_{rad}}\left(\frac{a_0}{a}\right)^2$$

$$a\dot{a} = \text{const} \Rightarrow \int_{t_0}^{t}\frac{da}{dt}a(t)\,dt = \frac{a^2 - a_0^2}{2} = \text{const}\cdot(t - t_0)$$

$$a(t) = \sqrt{c(t-t_0) + a_0^2}$$

$$\approx (ct)^{1/2} \quad ; \text{ for } a \gg a_0,\; t \gg t_0$$

$\Rightarrow T \propto t^{-1/2}$

- The evolution of H as a function of t in matter era can be obtained in a similar way:

$$H^2 = \frac{8\pi G}{3}\rho_{matter} \cong \frac{8\pi G}{3}\Omega^0_m\left(\frac{a_0}{a}\right)^3 \; ; \; \frac{\dot{a}}{a}\cdot a^{3/2} = \text{const} \; ; \int da\sqrt{a} = \left(a^{3/2} - a_0^{3/2}\right)\frac{2}{3} = c(t - t_0)$$

$\Rightarrow a \simeq (ct)^{2/3}$.

- $z_{eq} = 3400$, $T_{eq} = 0.75\,eV = 9180\,K$; eq matter with Λ: $z^{M\Lambda} \cong 0.3$; $T^{M\Lambda} \simeq 9\,K \simeq 7\cdot 10^{-4}\,eV$

Rates & Boltzmann equation: $1\,K = 8.6\cdot 10^{-5}\,K$

The distribution function $f(t,p)$ can be out of equilibrium and can evolve in time depending on the interactions happening between particles.
It is convenient to track the evolution of its integral, $n \sim \int f\, dp$
In absence of interactions, the number density of a species dilutes as the volume expands:

$$(*)\quad \dot{n} + 3H\,n = 0 \quad \Rightarrow \quad n \propto a^{-3} \quad \text{(Conservation of particle number)}$$

However, in the presence of scatterings we can have changes in particle number

$a + b \longrightarrow c + d$ (e.g. we will see recombination $e^- + p \to H + \gamma$ and Thompson scattering $e^- + \gamma \to e^- + \gamma$).

particle type

These collisions are described by a CROSS SECTION σ and the rate of the process is: $\sigma\,|v|\,n_a n_b$

avg. velocity

number density "squared": we need two particles to collide

This rate changes the RHS of $(*)$: we'd have

$$\dot{n}_a + 3H n_a = -\langle\sigma|v|\rangle\, n_a n_b + \beta\, n_c n_d$$

particle x is being annihilated

effect of inverse process

However, the inverse process $c+d \to a+b$ can happen as well.
DETAILED BALANCE: at equilibrium the number density remains unaffected since the inverse process is as likely. This means at equilibrium the RHS vanishes

$$-\langle\sigma|v|\rangle\, n_a^{eq} n_b^{eq} + \beta\, n_c^{eq} n_d^{eq} = 0$$

$$\Rightarrow \beta = \left(\frac{n_a n_b}{n_c n_d}\right)_{eq} \langle\sigma|v|\rangle$$

$$(++)\qquad \dot{n}_a + 3H n_a = -\langle\sigma|v|\rangle\left[n_a n_b - n_c n_d\left(\frac{n_a n_b}{n_c n_d}\right)\Big|_{eq}\right]$$

EXERCISE:

i) Rewrite LHS of $(++)$: show it can be written as $\frac{1}{a^3}\frac{d}{dt}(a^3 n_a)$

ii) Introduce $N_i \equiv n_i / s$, where $s \equiv$ entropy density. From previous sections recall that $s = s_0\left(\frac{a_0}{a}\right)^3$ for some constants s_0, a_0.

Show $\frac{1}{a^3}\frac{d}{dt}(n_a a^3) = \frac{s_0}{a^3} H \frac{dN_a}{d\log a}$

iii) Define $\Gamma \equiv \langle\sigma|v|\rangle n_b$. Show, using the previous steps, that:

$$\frac{d\log N_a}{d\log a} = -\frac{\Gamma}{H}\left[1 - \left(\frac{N_a N_b}{N_c N_d}\right)_{eq}\left(\frac{N_c N_d}{N_a N_c}\right)\right]$$

$$(++) \qquad \frac{d\log N_a}{d\log a} = \underbrace{-\frac{\Gamma}{H}}_{\text{efficiency}} \left[1 - \underbrace{\left(\frac{N_a N_b}{N_c N_d}\right)\Big|_{eq} \left(\frac{N_c N_d}{N_a N_b}\right)}_{\text{deviation from equilibrium}} \right]$$

- In practice what matters is the efficiency: consider that b, c, d are at equilibrium and $N_a \neq N_a|_{eq}$ (for example $N_a \gg N_a|_{eq}$). Consider large efficiency $\Gamma/H \gg 1$. The equation becomes approximatively

$$\frac{d}{d\log a}\log N_a \simeq -\frac{\Gamma}{H} \qquad \Rightarrow \text{ Na decreases towards } N_a^{eq}.$$

For small efficiency $\Gamma/H \ll 1$ the reactions don't bring Na to the equilibrium value.

This makes sense: $\Gamma > H$ means that we have many collisions in the time-scale of the expansion. Particles have time to thermalize before the universe expands.
For $\Gamma < H$ we have the opposite: expansion is faster than the rate of scattering.

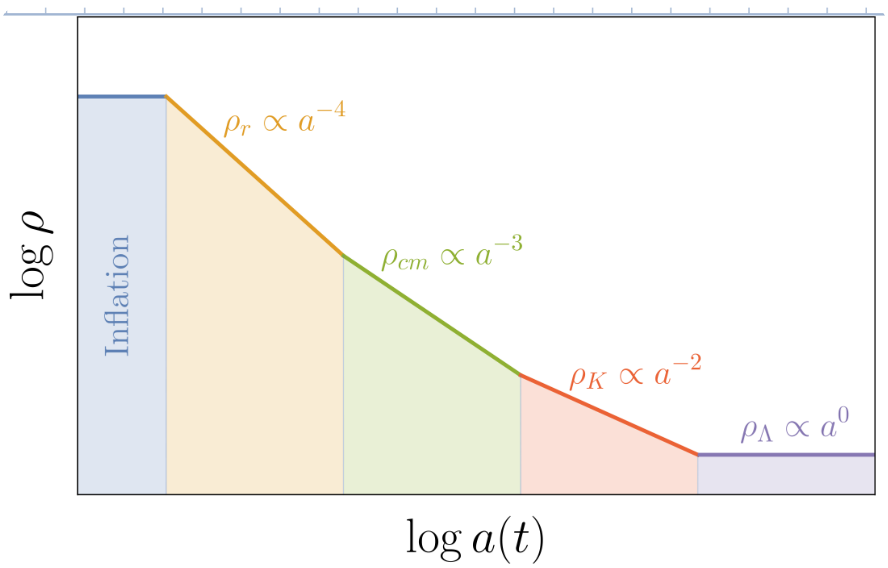
Brief History of the Universe

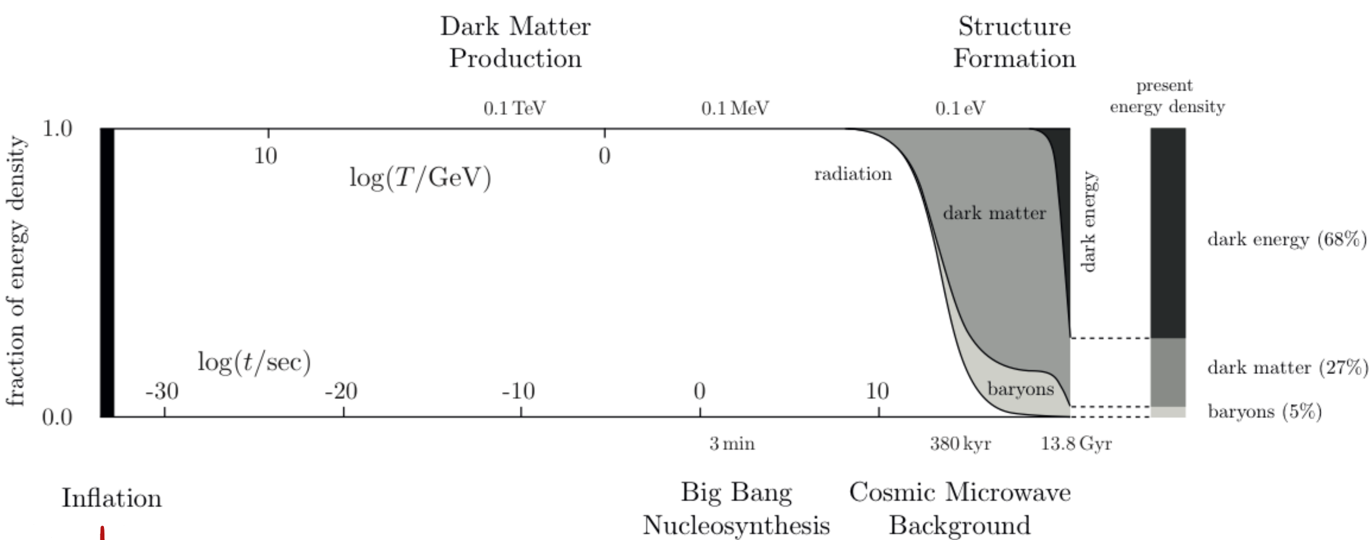
From previous formulas we have seen that $a(t)$ is larger in the past, and the behaviour of the energy components is:

RADIATION : $\rho \propto a^{-4}$; MATTER (Dark Matter & Bayons) $\rho \propto a^{-3}$

CURVATURE : $\rho \propto a^{-2}$; DARK ENERGY: $\rho \propto$ const

Evolution of $\Omega_R, \Omega_k, \Omega_m, \Omega_\Lambda$ during hystory :





[Baumann, Lecture notes]

last

Early phase of "Dark-Energy" expansion.
Can explain why curvature is small and "size" of inhomogeneities in the universe.

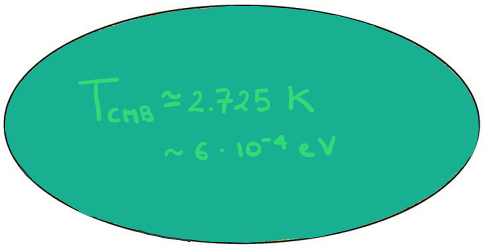
this lecture

The universe becomes transparent: photons are free to propagate.
Also, the universe becomes neutral (no ions).

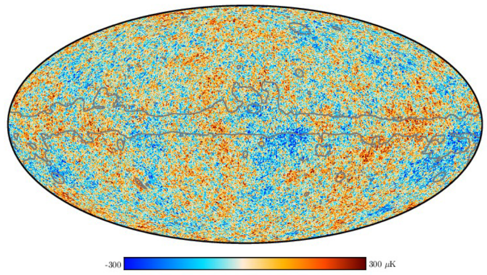
Event	time t	redshift z	temperature T
Inflation	10^{-34} s (?)	–	–
Baryogenesis	?	?	?
EW phase transition	20 ps	10^{15}	100 GeV
QCD phase transition	20 μs	10^{12}	150 MeV
Dark matter freeze-out	?	?	?
Neutrino decoupling	1 s	6×10^9	1 MeV
Electron-positron annihilation	6 s	2×10^9	500 keV
Big Bang nucleosynthesis	3 min	4×10^8	100 keV
Matter-radiation equality	60 kyr	3400	0.75 eV
Recombination	260–380 kyr	1100–1400	0.26–0.33 eV
Photon decoupling	380 kyr	1000–1200	0.23–0.28 eV
Reionization	100–400 Myr	11–30	2.6–7.0 meV
Dark energy-matter equality	9 Gyr	0.4	0.33 meV
Present	13.8 Gyr	0	0.24 meV

[Baumann, Lecture notes]

Cosmic Microwave Background



homogeneous part



anisotropies

Homogeneous microwave radiation: thermal spectrum extremely homogeneous of photons ($T_{CMB} \sim 2.7$ K).

Discovered in 1964 by Penzias & Wilson (Nobel in '78)

The CMB is made out of relic photons that in the early universe where at thermal equilibrium with electrons and protons.
e^+ and p formed a hydrogen plasma glowing at a homogeneous temperature. Photons where scattering with electrons (and hydrogen) so they where "trapped": the universe was an opaque cloud.
The expansion of the universe brings the temperature of the plasma down.
Hence, at some point e^+ & p combine to form neutral hydrogen H ($e^- + p \leftrightarrow H + \gamma$) and photons don't have enough energy to ionize H anymore (**RECOMBINATION**).
$\Rightarrow$ Universe becomes neutral.

Also, before recombination, photons & e^- where scattering between each other $e^- + \gamma \longleftrightarrow e^- + \gamma$ via THOMPSON SCATTERING.
Immediately after recombination the number of e^- around drops drastically: photons don't scatter anymore and are free to propagate $\Rightarrow$ The universe becomes transparent. The photons we see in the CMB are the ones that last scattered.
(**PHOTON DECOUPLING**).

- The radiation we see has $T_{CMB} \simeq 2.7$ K and is thermal.
$\rho_\gamma = \sigma T_{CMB}^4 \simeq 0.26$ MeV m^{-3} ; $\Omega_r^{today} \sim 5 \cdot 10^{-5}$ critical rad. density today.
$n_\gamma \sim \frac{\rho_\gamma}{T_{CMB}} \sim 400$ photons cm^{-3}

Baryons are more abundant today: $\Omega_b \simeq 0.04$; $\rho_b = \frac{3H_0^2}{8\pi G} \Omega_b \simeq 210$ MeV m^{-3}
(from experiments)
(* check these numbers)

baryons ~ hydrogen . $m_H \sim m_p \sim 1$ GeV (heavy) ; $n_b = \frac{\rho_b}{m_p} \simeq 0.22$ m^{-3}

BARYON - PHOTON RATIO : $\eta = \frac{n_b^0}{n_\gamma^0} \simeq 5 \cdot 10^{-10}$ (observational fact)

There are much more photons compared to atoms. This ratio is constant back to CMB. Since $\eta \ll 1$, at CMB it is more difficult to form H (takes longer)...

Recombination

The BINDING ENERGY of H is $B_H \simeq 13.6\ eV$. We'd expect that RECOMBINATION happens around $T_\gamma \simeq B_H$. However this turns out to be not correct!
Due to the large number of photons we need lower temperatures.
A proper calculation gives:

$$T_{REC.} \simeq \frac{B_H}{42} \simeq 0.3\ eV \simeq 3700\ K \quad \text{as opposed to} \quad B_H \simeq 60'000\ K$$

From this value we can work out the redshift at recombination.
Soon after recombination photons travel freely, meaning they redshift like radiation, of course
$\rho_\gamma \propto T^4 \propto a^{-4} \Rightarrow T \propto a^{-1}$

scale factor today

$$\Rightarrow \frac{T_{REC}}{T_{CMB}} = \frac{a_0}{a(t_{rec})} = (1+z_{rec}) = \frac{3700\ K}{2.7\ K} \Rightarrow z_{rec} \simeq 1320$$

CMB temp. today

Decoupling

A process closely related to recombination.
Photons remain coupled with the plasma due to THOMSON SCATTERING:
$e^- + \gamma \leftrightarrow e^- + \gamma$

Recall that the Thomson cross-section $\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0 m_e c^2}\right)^2$

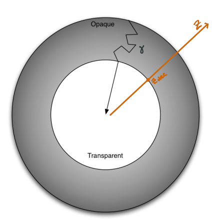
In natural units $[c = \hbar = k_B = 1]$ we get

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{4\pi m_e}\right)^2 = \frac{8\pi}{3}\frac{\alpha^2}{m_e^2} \quad ; \quad \alpha \equiv \frac{e^2}{4\pi}$$
$$\simeq 2\cdot 10^{-3}\ MeV^{-2}$$
$$\simeq 6.6\cdot 10^{-29}\ m^2$$

number density of electrons

- The rate of scatterings is $\Gamma \simeq n_e \sigma_T$
  Since electrons become bounded in hydrogen, n_e drops during recombination.
  $\Rightarrow \Gamma$ drops $\Rightarrow$ decoupling basically coincides with recombination

(*) The scatterings become not effective when Γ becomes smaller than the rate of the expansion: H.
From this equation one can in principle work out z_{dec}



$$\Rightarrow z_{dec} = 1100$$

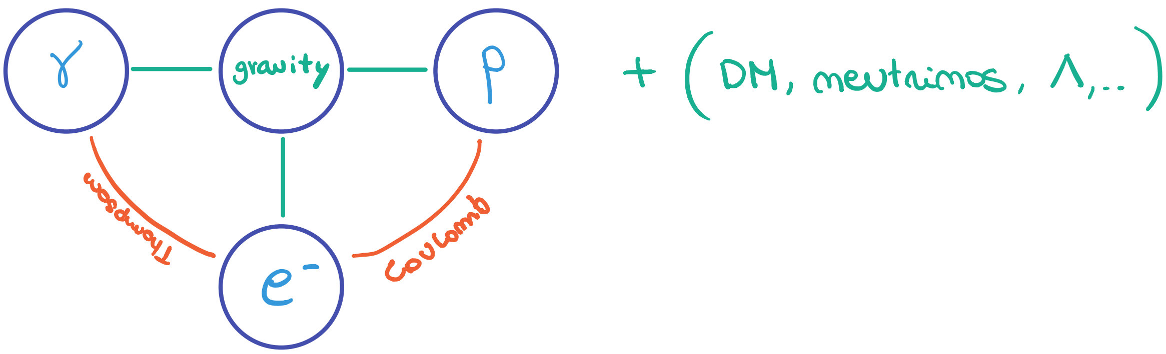
- Note that decoupling / recombination happen when we are MATTER DOMINATED.
  EXERCISE: find $z_{equality}$ = redshift of matter - radiation equality
  i.e. when $\rho_r = \rho_m$

$$\Omega_m^0 (1+z_{eq})^3 = \Omega_r^0 (1+z_{eq})^4 \Rightarrow (1+z_{eq}) = \frac{\Omega_m^0}{\Omega_r^0} \simeq 3365 \pm 44$$

CMB Anisotropies

Fluctuations in the uniform temperature are small : $\frac{\delta T}{T} \simeq \frac{300\,\mu K}{2.7\,K} \sim 10^{-4}$
however they contain a lot of informations.
CMB fluct. are a great probe :
- they are small / linear
- can be studied with known atomic physics

These fluctuations correspond to density fluctuations in the plasma of PHOTONS, ELECTRONS & PROTONS. These components interact via Thompson and Coulomb scattering, but are also affected by gravity.



We can describe the plasma as a fluid : baryon-photon fluid.
Temperature fluctuations $\longleftrightarrow$ density fluctuations of photons
$\longleftrightarrow$ " " " baryon-photon fluid.
$\longleftrightarrow$ sound waves

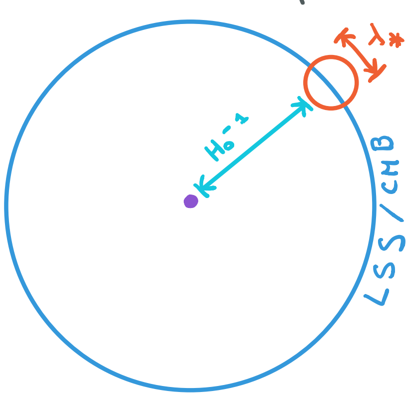
$\Rightarrow$ Sound waves are characterized by a SPEED OF SOUND $c_s \lesssim c$.
(For photon fluid $c_s^2 \simeq \frac{1}{3} c^2$).

$$\ddot{\Theta} - c_s^2 \nabla^2 \Theta = 0$$

(* expansion of the universe needs to be accounted for. Here we neglect it for simplicity).

Linear fluctuations $\longrightarrow$ Fourier analysis $\longrightarrow$ $\Theta(k,t) \sim A_k \cos(k c_s t)$

- Peaks : waves that, at recombination, are at maxima of $\cos(k c_s t)$ are enhanced (peaks in temperature). If $t_* = t_{rec}$ then
peaks : $k_* = \frac{m\pi}{c_s t_*} \simeq \frac{m\pi\sqrt{3}}{t_*}$; $\lambda_* \simeq \frac{1}{k_*}$
↳ special scale of the fluctuations : this gives a typical angular scale in the CMB (angular size of fluctuations)



MATTER ERA : $H^2 \propto \rho_m \propto a^{-3}$

$$\theta_* \simeq \overset{(+)}{\boxed{(1+z_{rec})}} \frac{\lambda_*}{H_0^{-1}} \sim (1+z_r)\frac{H_0}{H_{rec}} \simeq (1+z_{rec}) \frac{H_0}{H_0 \left[\Omega_m^0 (1+z)_{rec}^3\right]^{\frac{1}{2}}}$$

$$\simeq \frac{(1+z_{rec})}{\sqrt{\Omega_m^0}} (1+z_{rec})^{-3/2} = \frac{(1+z_{rec})^{-\frac{1}{2}}}{\sqrt{\Omega_m^0}}$$

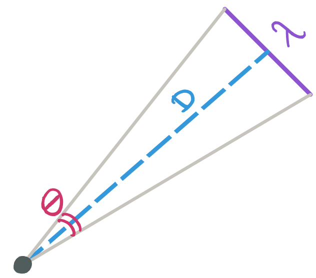
$$\simeq 0.03 \text{ radiants} \sim 2^\circ$$

$(\dagger) \equiv$ corrections due to the expansion, similar to the $\sim(1+z)$ in the expression of d_L (LUMINOSITY DISTANCE).

• ANGULAR DIAMETER DISTANCE:

Suppose we have an object whose size is known (e.g. λ_*) but its distance not.
What we usually measure is the angle θ subtended by this object in the sky.
In flat space, for far-away objects, the angle can be related to the distance D as:

$$\theta \approx \frac{\lambda}{D} \equiv \frac{\text{intrinsic size}}{\text{distance}}.$$



In curved FLRW space we are led to define the ANGULAR DIAMETER DISTANCE in analogous terms:

$$D_A \equiv \frac{\lambda}{\theta}$$

In FLRW geometry the physical size λ can be written as

$$\lambda = \theta \cdot \underbrace{r_e}_{\text{comoving distance}} \underbrace{a_e}_{\text{scale factor at emission}}$$

$$\Rightarrow D_A = r_e a_e = r_e a_0 \frac{a_e}{a_0} = \frac{a_0 r_e}{(1+z)}$$

$$\Rightarrow \theta = \frac{\lambda}{r_e a_e} = \frac{\lambda}{r_e a_0}\left(\frac{a_0}{a_e}\right) = \frac{\lambda(1+z)}{(r_e a_0)}$$ this explains $(\dagger)$.

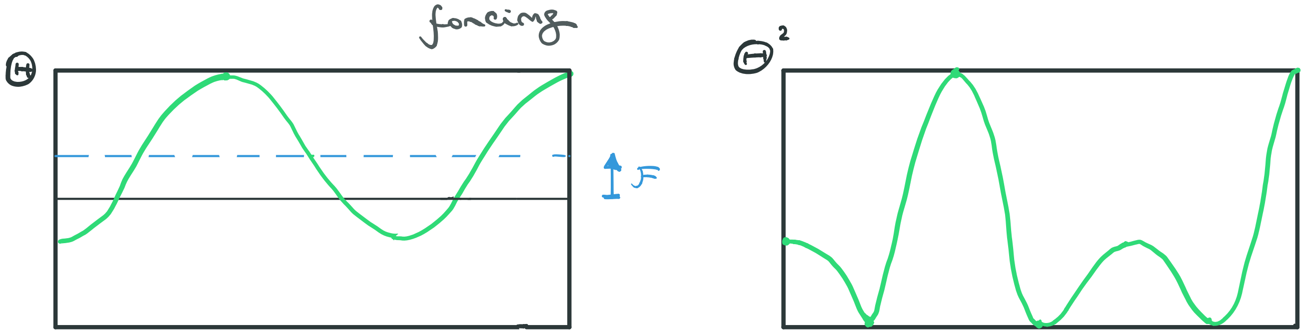
This size ultimately depends on c_{s*}: depends on cosmological parameters.

• The position of the first peak depends on the SPATIAL CURVATURE: as we have seen positive curvature leads to larger angles θ as compared to Euclidian space (for same size).
This means we could interpret $K>0$ as larger objects.
$K<0$ as smaller objects.

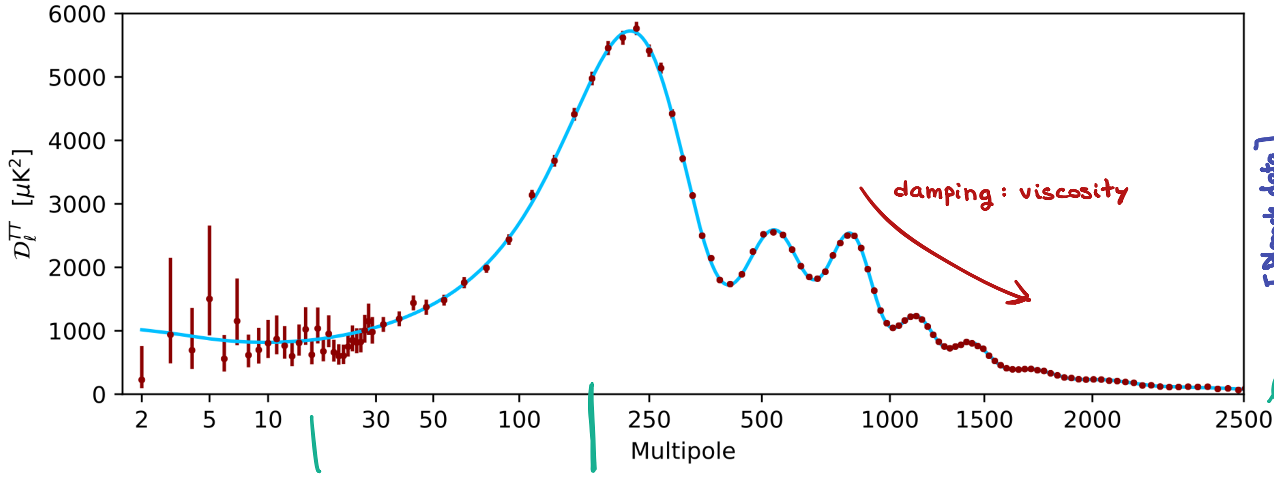
$\Rightarrow$ CMB probes curvature Ω_K today: strong evidence for $K \simeq 0$ with $\lesssim 1\%$ precision.

• Effect of baryons: more baryons imply a smaller c_s.
effectively this translates into a "forced" harmonic oscillator in Fourier space:

$$\ddot{\Theta} + c_s^2 k^2 \Theta = \mathcal{F}$$ $\longrightarrow$ shift due to the forcing $\longrightarrow$ Enhancement of even peaks.



CMB temperature correlation: $\left\langle \frac{\delta T}{T} \frac{\delta T}{T} \right\rangle \sim \Theta^2$



damping: viscosity

[Planck data]

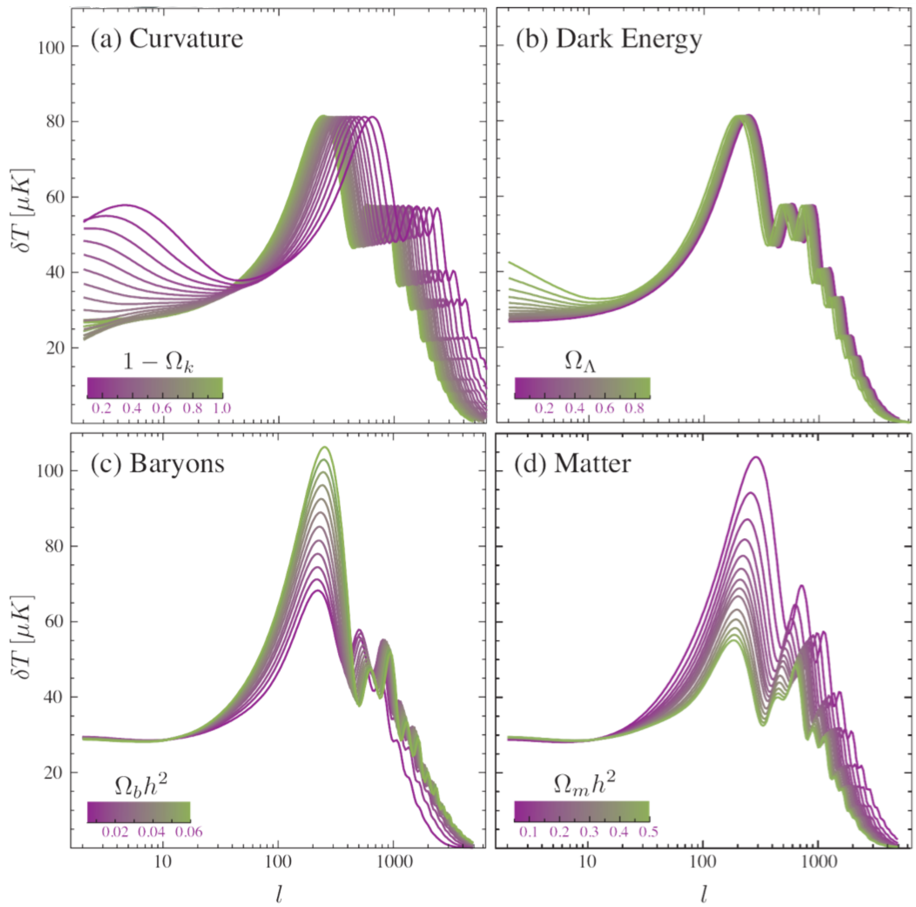
$\ell \simeq$ wave number

large angular scales: unaffected by baryon-photon oscillations.

First peak: $\ell \simeq 200$

$$\ell \sim \frac{1}{\theta}$$

Parameter Dependence: The CMB spectrum is the result of many processes, which in turn depend on different cosmological parameters. To better understand how these parameters affect the shape of the spectrum let's look at these plots:



[Baumann, Lecture notes]

a) curvature Ω_k : Recall that $\Omega_k = \frac{-K}{a^2H^2}$. By varying Ω_k from 0 to 1 we are adding negative curvature to "space". Negative curvature makes angles smaller, so we would see a shift in the CMB peaks to smaller angles (i.e. larger ℓ). The measure of the position of the peaks give informations / constraints on Ω_k.

b) dark energy Ω_Λ : CMB is formed in matter dominated era, when Λ was negligible. However, Λ affects the propagation of photons from the last-scattering surface and us. Therefore, the effect of Λ is mainly at large scales. (Integrated Sachs-Wolfe effect).

c) baryons Ω_b : The effect of baryons is to change the forcing term $\mathcal{F}$ in the equation for Θ. This means that the relative heights of the peaks change by changing Ω_b. This clearly seen in fig. (c).

d) matter Ω_m : By decreasing Ω_m we are moving matter-radiation equality closer to the CMB (at $z_{dec} = 1100$). As a consequence of this the forcing term becomes more efficient and the whole spectrum is boosted up.

EXERCISE: Neutrino decoupling & Neutrino temperature.

An interesting case of study are NEUTRINOS. Neutrinos are very light particles but massive nonetheless ($\sum_i m_{\nu_i} < 0.3$ eV). Since they are very light they were also relativistic for a very long time (some of them might still be relativistic).
They interact very weakly. These facts imply that they decoupled from the "thermal bath" while still in the relativistic limit (HOT RELICS)

Other features:
- ν are fermions (like electrons & protons) and have an associated anti-particle $\bar{\nu}$.
- they interact via WEAK INTERACTIONS. This means they stop being in thermal equilibrium before being able to annihilate into other particles $\Rightarrow$ their density is higher than the one of p & e^-.
- when e^- & e^+ annihilate they produce photons γ, not neutrinos this means that energy is "added" to γs, not to νs.

$$\Rightarrow T_\nu \text{ is smaller than } T_\gamma : \quad T_\nu = \left(\frac{4}{11}\right)^{\frac{1}{3}} T_{CMB}.$$

- When do ν decouple (what temperature)?

They interact via the interactions $\nu_e + \bar{\nu}_e \leftrightarrow e^+ + e^-$, $e^\pm + \nu_e \leftrightarrow e^\pm + \nu_e$
$e^\pm + \bar{\nu}_e \leftrightarrow e^\pm + \bar{\nu}_e$

whereas μ, τ neutrinos only react with $e^\pm + \nu_{\mu,\tau} \longleftrightarrow e^\pm + \nu_{\mu,\tau}$

These are weak interactions: the typical cross-section $\sigma \sim G_F^2 T^2$
$(G_F = \alpha_W / M_W \sim 10^{-5}\ \text{GeV}^{-2} \quad (\alpha_W \simeq 1/30))$

$$\Gamma \simeq \underset{\substack{\downarrow \\ \text{relativistic}}}{n} \sigma \cdot v \simeq G_F^2 T^5 \ ; \quad \frac{\Gamma}{H} \simeq \frac{G_F^2 T^5}{T^2 G_N^{1/2}} \simeq \left(\frac{T}{\text{MeV}}\right)^3 \overset{?}{\lessgtr} 1$$

This estimate shows that neutrinos decouple at temperatures $T \simeq 1$ MeV.

Inflation

The standard "BIG BANG" picture (from radiation era) works very well, but it faces some conceptual issues. Inflation is thought to be an era preceding radiation, and tries to alleviate these problems. Additionally, the "inflationary paradigm" is able to explain the features of the fluctuations we see in CMB and LSS.

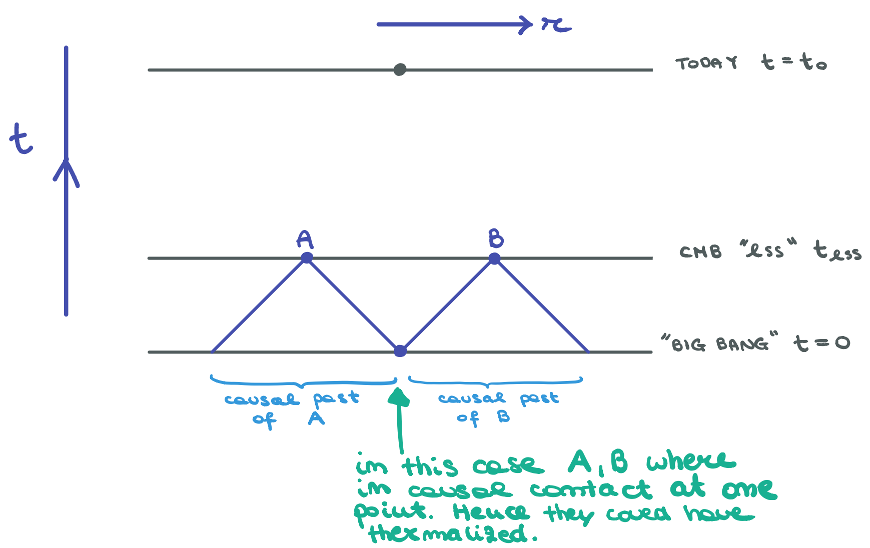
First, we have a look at the problems we are facing...

① **HORIZON PROBLEM:** we have seen that the CMB has a homogeneous temperature in all sky directions (with fluctuations $\sim 10^{-5}$).

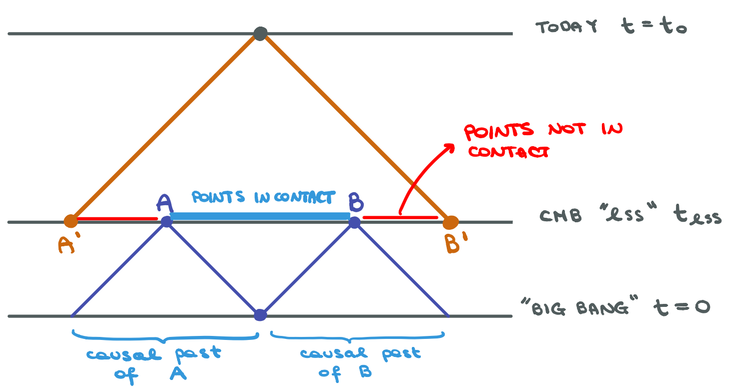
Information can travel at most at the speed of light. In FLRW it means we have to look at the light-cones:

$$dr = \frac{dt}{a(t)} \qquad (+)$$

In order to explain this homogeneity (and without assuming special initial conditions) we would like to require that different points in the CMB where in causal contact in the past. In this way they could have thermalized together.



However, we find that not all the points of the CMB we see where in causal contact



POINTS IN CAUSAL CONTACT: The comoving distance A-B can be obtained by integrating formula (+):

$$R_{causal} = R_{AB} = \int_{t=0}^{t=t_{LSS}} \frac{dt'}{a(t')} \simeq \frac{1}{aH}\Big|_{LSS}$$

[We assume we are in rad. era from $t=0$ until t_{LSS}. Thus $a(t) = t^{1/2} a_0$, $H(t) = \frac{1}{2t} \propto a^{-2}$

The integral becomes

$$R_{causal} = \int_0^{t_{LSS}} \frac{dt'}{a_0 (t')^{1/2}} = \frac{1}{a_0} 2\left(t_{LSS}^{1/2}\right) = \frac{2\, t_{LSS}}{a(t_{LSS})} = \frac{1}{a(t_{LSS})\, H(t_{LSS})}$$]

COMOVING RADIUS OF LSS: This is the distance A'B', and can be obtained in a similar way

$$R_{LSS} = R_{A'B'} = \int_{t_{LSS}}^{t_0} \frac{dt'}{a(t')} \simeq \frac{2}{aH}\Big|_{t_0}$$

[Here we are in matter era: $a(t) = a_0 t^{2/3}$, $H(t) = \frac{2}{3t} \propto a^{-3/2}$

$$R_{LSS} = \int_{t_{LSS}}^{t_0} \frac{dt'}{a_0 (t')^{2/3}} = \frac{1}{a_0} 3\left(t_0^{1/3} - t_{LSS}^{1/3}\right) \underset{t_0 \gg t_{LSS}}{\simeq} \frac{3\, t_0^{1/3}}{a_0} = \frac{3}{a_0 t_0^{2/3}}\, t_0 = \frac{2}{(aH)}\Big|_{t_0}$$]

• The angular size in the sky of causally-connected regions is (in radiants)

$$\frac{R_{causal}}{R_{LSS}} \underset{\text{ignoring factors of 2}}{\simeq} \frac{(aH)_0}{(aH)_{LSS}} \underset{\text{matter era}}{\sim} \frac{(a \cdot a^{-3/2})_0}{(a \cdot a^{-3/2})_{LSS}} = \left(\frac{a_{LSS}}{a_0}\right)^{1/2} = \frac{1}{\sqrt{1+z_{LSS}}} \simeq \frac{1}{\sqrt{1100}} \sim \frac{1}{30}$$

This corresponds to $\sim 1°$: basically everything is causally disconnected!

EXERCISE: Confirm this with a more precise calculation (numerically)

i) The precise formula for R between two times can be written in terms of z (recall d_L)

$$R(z_1, z_2) = \int_{t_1}^{t_2} \frac{dt'}{a(t')} = \int_{a_1}^{a_2} \frac{da'}{a'} \frac{dt}{da} = \int_{a_1}^{a_2} \frac{\dot{a}}{\dot{a}} \frac{1}{(a')^2} da' = \int_{a_1}^{a_2} \frac{da'}{(a')^2} \frac{1}{H} \quad ; \quad a = \frac{a_0}{(1+z)}, \ \frac{da}{a_0} = \frac{(-1)}{(1+z)^2} dz$$

$$= \frac{1}{H_0 a_0} \int_{z_2}^{z_1} \frac{dz}{[\Omega_\Lambda + \Omega_m^0 (1+z)^3 + \Omega_r^0 (1+z)^4]^{1/2}} \frac{1}{(1+z)^2} \quad ; \quad \Omega_\Lambda^0 \simeq 0.7 \, ; \, \Omega_M^0 \simeq 0.3$$

$$\Omega_r^0 \simeq 8.4 \cdot 10^{-5}$$

ii) Evaluate numerically $R_{causal} = R(\infty, 1100)$; $R_{LSS} = R(1100, 0)$

verify that $\frac{R_{causal}}{R_{LSS}} \simeq \frac{1}{50}$

② CURVATURE / FLATNESS PROBLEM :

Recall how K enters in the Friedmann equation:

$$\frac{3}{8\pi G_N} H^2 = \rho - \frac{3}{8\pi G_N} \frac{K}{a^2}$$

The curvature today is small, meaning that $\left| \frac{3K}{8\pi G_N a^2} \right| \Big/ \frac{3 H_0^2}{8\pi G_N} \ll 1$

i.e. $\Omega_K = \frac{-K}{(aH)^2}$ is small.

However, if it is small now it needs to be tiny in the past :

- $(aH) \propto a \cdot a^{-3/2} = a^{-1/2}$
- $\Omega_K \propto a$ so it grows at time increases : it was smaller in the past.

To obtain the value we see today we need tiny initial conditions for Ω_K. This seems very unnatural.

SOLUTION: postulate initial acceleration $\ddot{a} > 0$

If we have an era where the universe accelerates at the beginning (for example with a cosmological constant) then the integral R_{causal} becomes dominated by early times:

$$R_{causal} = \int_{t_{init}}^{t_{LSS}} \frac{dt'}{a(t')} = \int_{a_{in}}^{a_{LSS}} \frac{d\tilde{a}}{\tilde{a}\,\dot{\tilde{a}}} \,)$$

If $\dot{\tilde{a}}$ is not there : $R \sim \log \frac{a_{LSS}}{a_{in}}$. Depending on the behaviour of $\dot{a}$ we have that the integral diverges/is dominated at early or late times

If we expand around $a=0$: $\dot{a} \propto a^p \Rightarrow \int \frac{d\tilde{a}}{\tilde{a}^{(p+1)}}$ if $p>0$ diverges at early times; if $p<0$ diverges at late times

- For RD : $a \propto t^{1/2}$; $\dot{a} \propto \frac{1}{t^{1/2}} \sim \frac{1}{a}$
- For MD : $a \propto t^{2/3}$; $\dot{a} \propto \frac{1}{t^{1/3}} \sim \frac{1}{a^{1/2}}$

$\Bigg\} \Rightarrow p<0$ and early times do not count

- To solve horizon problem we need a phase of the universe with $p>0 \Rightarrow \ddot{a}>0$

$$\boxed{\text{INFLATION} \equiv \ddot{a} > 0 \text{ before RD}}$$

EXERCISE: Consider inflation as a Λ-dominated era: $P=-\rho$ $(\omega=-1)$
Verify the horizon problem can be solved: compute explicitly $\frac{R_{causal}}{R_{LSS}}$

- For $P=-\rho \Rightarrow \dot{H}=0 \Rightarrow a = a_{rad}\cdot e^{H_{inf}(t-t_{rad})}$
$H(t)=H_{inf}$

- The universe starts in inflation then goes to RD and MD.

$$R_{causal} = \int_{t_{inf}}^{t_{LSS}} \frac{dt'}{a(t')} = \int_{t_{inf}}^{t_{rad}} \frac{dt'}{a(t')} + \underbrace{\int_{t_{rad}}^{t_{LSS}} \frac{dt'}{a(t')}}_{\frac{1}{(aH)|_{LSS}}} \simeq R_{inf} + \frac{1}{(aH)|_{LSS}}$$

$$R_{inf} \equiv \int_{t_{inf}}^{t_{rad}} \frac{dt'}{a(t')} = \int_{t_{inf}}^{t_{rad}} \frac{dt'}{a_{rad}}\, e^{H_{inf}(t-t_{rad})} = \frac{1}{a_{rad} H_{inf}}\left(e^N - 1\right)$$

$$N \equiv H_{inf}\cdot(t_{rad}-t_{inf}) \equiv H_{inf} \underbrace{\Delta t}_{\text{duration of inflation}}$$

- $N \equiv$ measure of duration of inflation.

- We can solve the horizon problem if $e^N \gg 1$:

$$R_{causal} \cong \frac{1}{(a_{rad} \underbrace{H_{inf}}_{=H_{rad}})}\left(e^N - 1\right) + \frac{1}{(aH)|_{LSS}} \simeq \frac{e^N}{(aH)|_{rad}} + \frac{1}{(aH)|_{LSS}}$$

$$R_{LSS} \simeq \frac{1}{(a_0 H_0)} \quad \text{as before.}$$

$$\frac{R_{causal}}{R_{LSS}} \simeq \frac{a_0 H_0}{(aH)|_{LSS}}\left(1 + e^N \cdot \frac{(aH)|_{LSS}}{(aH)|_{rad}}\right) \simeq \frac{1}{\sqrt{1100}}\left(1 + e^N\, \frac{(aH)_{LSS}}{(aH)_{rad}}\right)$$

- $$\frac{(aH)_{LSS}}{(aH)_{rad}} \underset{\substack{\text{rad era} \\ H\sim 1/a^2 \\ a \sim 1/T}}{\simeq} \frac{T_{LSS}}{T_{rad}}$$

- We are certain that radiation era extends at least back to nucleosynthesis
$T_{rad} > 10^{10}$ Kelvin
$T_{LSS} \simeq 10^3$ Kelvin

$$\frac{R_{causal}}{R_{LSS}} \sim \frac{1}{30}\left(1 + \frac{10^3}{10^{10}}\, e^N\right)$$

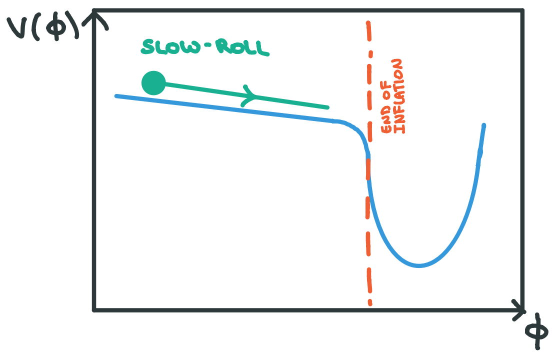
. Now we can make the ratio of order, say, $\sim \mathcal{O}(1)$ by taking:

$$10^{-7} e^N \simeq (30-1) \Rightarrow e^N \simeq 10^7 \cdot 30 \sim 10^8 \Rightarrow N \sim \ln(10^8) \sim 20.$$

(If we require inflation to happen at $T_{rad} \sim 10^{29}$K (GUT scale) then $N \sim 60$).

Slow-roll inflation

In practice one needs to specify the type of "matter" responsible for the acceleration.
The most robust/accepted model is slow-roll inflation:
SCALAR FIELD ϕ (like the Higgs particle) slowly rolling down a potential



The "inflaton" $\phi(t)$ moves slowly since its potential $V(\phi)$ is initially very flat. Also, Hubble expansion gives friction / energy dilution.

$$\ddot{\phi}(t) + 3H(t)\dot{\phi}(t) + V'(\phi) = 0$$

(DAMPED OSCILLATOR: $\ddot{\phi}$ smaller than the other terms).

The cosmological constant / acceleration is given by $V(\phi) \sim \Lambda$:

$$H^2 = \frac{8\pi G_N}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) \approx \frac{8\pi G_N}{3} V(\phi)$$

(since ϕ moves slowly and $V'(\phi)$ is small).

Features:

- Slow-roll inflation has a mechanism to END inflation: $\approx$ when $\ddot{\phi}$ becomes important (REHEATING: ϕ decays into standard particles & we enter in radiation era).

- It explains the size / features of inhomogeneities we see (for example in the CMB)
  Idea: ϕ has at least quantum fluctuations:

$$\phi(t,x) = \underbrace{\phi(t)}_{\text{homogeneous slow-rolling solution}} + \underbrace{\delta\phi(t,x)}_{\text{quantum initial fluctuations}}$$

the quantum fluctuations $\delta\phi$ are not deterministic, but correlations are dictated by quantum mechanics ($\Delta p \cdot \Delta x \geq \frac{\hbar}{2}$).

It is "easy" to estimate $\langle \delta\phi(t,x)^2 \rangle$. $H(t) \approx H_{inf}$ is almost constant and $a(t) \sim e^{Ht}$
Expansion "stretches" the Fourier modes (waves) so much that they become slowly-varying.
=> $\langle \delta\phi^2 \rangle$ does not depend on (t,x). By dimensional analysis it has to scale as H^2 (H is the only dimensionful parameter relevant for $\delta\phi$).

$$\boxed{\langle \delta\phi^2(t,x) \rangle \sim H^2}$$

Often called "SCALE INVARIANT SPECTRUM". Is consistent with what we see in CMB.

ADDITIONAL MATERIAL:

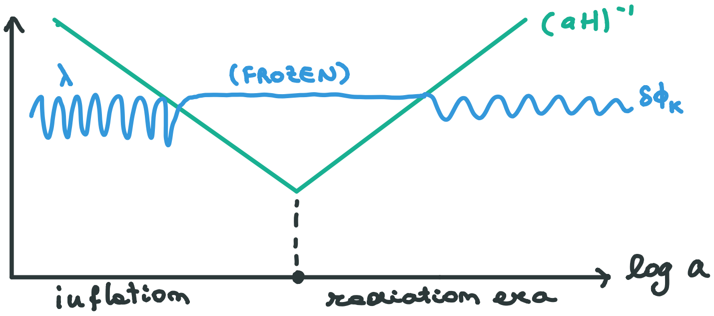
- Now we give a very brief recap of the features of inflation, so as to get accustomed to the terminology cosmologists use...

SCALE INVARIANCE: We can get an intuition for why the spectrum is "scale invariant" by understanding the evolution of a generic Fourier mode $\delta\phi_k(t)$ with wavelength $\lambda = 2\pi/k$.
We get different behaviours for the regimes $\frac{k}{aH} \ll 1$ and $\frac{k}{aH} \gg 1$.

For $\frac{k}{aH} \gg 1$ the (physical) wavelength is much shorter than the size of the universe, and effectively it is like being in flat space: the Fourier modes oscillate in time.

For $\frac{k}{aH} \ll 1$ we reach a point where λ_{phys} becomes larger than the universe. The mode does not evolve and FREEZES. (Recall the behaviour of reactions when $\Gamma/H \ll 1$.)

· Additionally, because we have acceleration $\ddot{a} > 0$ we also have that $(aH)^{-1}$ decreases during inflation. We can thus plot the behaviour of a mode $\lambda_{comoving}$:



When a mode freezes we say it "exits the horizon".

In practice, we can obtain the spectrum of fluctuation at the horizon crossing since after that point modes don't evolve.
Such modes will start evolving again once they reenter the horizon during radiation (or matter) era.

To find the spectrum at horizon crossing we need to evaluate $\langle \delta\phi^2 \rangle \sim H$ when $\frac{k}{aH} = 1$.
Since during inflation $H \approx$ const we have that $\langle \delta\phi^2 \rangle$ will have only a tiny time dependence $\Rightarrow$ tiny k dependence. This is what is meant by scale invariance.

Usually the spectrum in Fourier space, $\Delta_{(s)}$, is parametrized as follows:

$$\Delta_{(s)} = A_s \left(\frac{k}{k_*}\right)^{1-n_s}$$

$n_s \equiv$ SPECTRAL INDEX

, where $k_* = 0.05\ \text{Mpc}^{-1}$ is an arbitrary reference scale.

This (scalar) power spectrum is exactly scale invariant if $n_s = 1$: $\Delta_{(s)}$ becomes k-independent.
Slow-roll inflation predicts tiny deviations from scale invariance, and this is confirmed experimentally. Indeed Planck measured $n_s \simeq 0.96$. What it is also measured is:

$$A_s \simeq 2 \cdot 10^{-9} \quad \text{(amplitude of fluctuations)}$$

· Inflation leads to the production of primordial gravitational waves $h(t, x)$.
GW's are produced in a similar fashion: they start quantum mechanical and subsequently "grow" and freeze when exiting the horizon.

Their spectrum $\Delta(t)$ is parametrized similarly:

$$\Delta(t) = A_t \left(\frac{k}{k_*}\right)^{n_t}$$

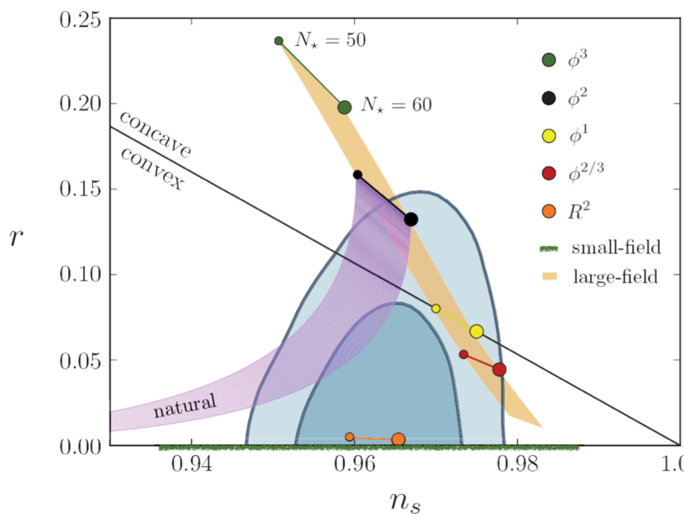
; (t) stands for "tensor modes".
Note that n_t has a different definition compared to n_s.

$A_t \equiv$ amplitude of GW fluctuations.

A major experimental challenge is to measure tensor modes: they leave an imprint in the polarization of photons in the CMB. When people talk about measuring primordial GW's they usually mean measuring the so-called "B-modes" of the CMB (particular type of polarization patterns of relic photons).

The parameter that experimentalists are after is $r \equiv \frac{A(t)}{A(s)}$ = TENSOR-TO-SCALAR RATIO which measures the power in GW compared with the power in scalar (inflaton) fluctuations.

So far Planck obtained a very precise measurement of n_s, but for r we only have an upper bound.



The blue region is the one favoured by Planck data.
Lines and colored regions refer to the predictions of different inflationary models.
Some of them are already ruled out.

- The detection of primordial gravitational waves be a very compelling argument in favour of slow-roll inflation.
  It would also give the a value for the ENERGY SCALE at which inflation took place. This would be a very strong indication for the existence of unknown physics beyond the standard model of particle physics.