Lecture 2 - The geodesic motion around a Schwarzschild BH

Geodesic motion ~~...~~ is described by

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \qquad \lambda \begin{cases} \to \text{proper time for massive particles} \\ \to \text{affine parameter for massless particles} \end{cases}$$

$\dot{x}^\mu = dx^\mu/d\lambda$

It comes from a Lagrangian

$$\mathcal{L}(x^\alpha, dx^\alpha/d\lambda) = \frac{1}{2} g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \quad \longrightarrow \quad \frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial(dx^\alpha/d\lambda)} = 0$$

For Schwarzschild one gets

$$\mathcal{L} = \frac{1}{2}\left[-\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-\frac{2M}{r}} + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right]$$

$p^\alpha = m u^\alpha = (\mathcal{E}, m\gamma v^i)$

- EQUATION FOR $\dot{t}$

$$0 = \cancel{\frac{\partial \mathcal{L}}{\partial t}} - \frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial \dot{t}} \Rightarrow \frac{d}{d\lambda}\left[\left(1-\frac{2M}{r}\right)\dot{t}\right] = 0 \to \dot{t} = \frac{C_1}{1-2M/r}$$

- $C_1 = E \to$ Energy per unit mass for massive particles $\mathcal{E}/m$
- $C_1 = E \to$ Energy for massless particles

- EQUATION FOR $\dot{\varphi}$

$$0 = \cancel{\frac{\partial \mathcal{L}}{\partial \varphi}} - \frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial \dot{\varphi}} \Rightarrow \frac{d}{d\lambda}\left[r^2\sin^2\theta\,\dot{\varphi}\right] = 0 \to \dot{\varphi} = \frac{C_2}{r^2\sin^2\theta}$$

- $C_2 = L \to$ angular momentum per unit mass for massive particles
- $C_2 = L \to$ angular momentum for massless particles

- EQUATION FOR $\dot{\theta}$

$$0 = \frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial \dot{\theta}} \Rightarrow \frac{d}{d\lambda}(r^2\dot{\theta}) = r^2\sin\theta\cos\theta\,\dot{\varphi}^2 \Rightarrow \ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} + \sin\theta\cos\theta\,\dot{\varphi}$$

Spherical symmetry allows to choose the reference frame such that for the initial condition $\lambda = 0$, then $\theta = \pi/2$ and the three velocity $(\dot{r}, \dot{\theta}, \dot{\varphi})$ lies on the same plane ($\dot{\theta} = 0$ at $\lambda = 0$)

The Cauchy problem admits the only solution $\theta(\lambda) = \pi/2 \Rightarrow \dot{\theta} = 0$ always

Orbits are planar like in newtonian gravity!

- EQUATION FOR $\dot{r}$

1) Massive particle $u_\alpha u^\alpha = -1$

2) Massless particle $u_\alpha u^\alpha = 0$

$$\to -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-2M/r} + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2 = \begin{cases} -1 & m \neq 0 \\ 0 & m = 0 \end{cases}$$

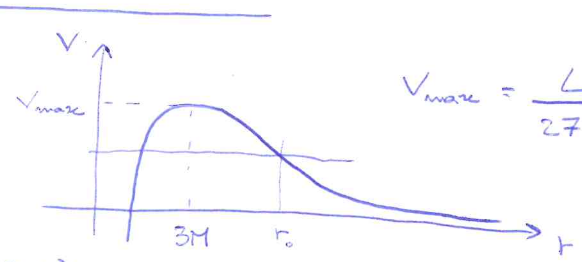
$$-\frac{E^2}{1-\frac{2M}{r}} + \frac{\dot{r}^2}{1-\frac{2M}{r}} + \frac{L^2}{r^2 \cancel{\sin\theta}} = \begin{cases} -1 \\ 0 \end{cases}$$

$$\to \dot{r}^2 + \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{r^2}\right) = E^2$$

$$\to \dot{r}^2 + \left(1-\frac{2M}{r}\right)\frac{L^2}{r^2} = E^2$$

MASSLESS CASE

$$\dot{r}^2 = E^2 - V(r) \qquad V = \frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)$$



$V_{max} = \frac{L^2}{27M^2}$

$\dot{r} < 0$ from $r = +\infty$

$$2\dot{r}\ddot{r} = -\frac{dV}{dr}\dot{r} \Rightarrow \ddot{r} = -\frac{1}{2}\frac{dV}{dr} \qquad \text{acceleration} \quad \ddot{r} = \frac{L^2}{r^3}\left(1-\frac{3M}{r}\right)$$

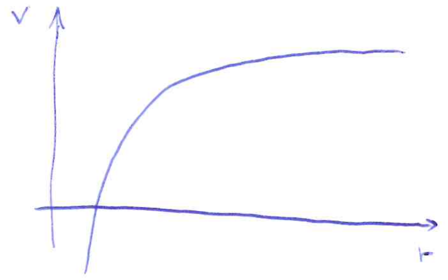
- $E^2 > V_{max} \longrightarrow$ particle will always fall in the body
- $E^2 = V_{max} \longrightarrow \dot{r} = 0$ at $r = 3M$, as well as $\ddot{r} = 0$. Circular orbit, unstable! Infall if displaced to $r < 3M$, escape to infinity in $r > 3M$
- $E^2 < V_{max} \longrightarrow r_0$ is a turning point $\longrightarrow$ Light is deflected back to infinity!

MASSIVE CASE

$$\dot{r}^2 = E^2 - V(r) \qquad V = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right)$$

$$V' = \frac{2M}{r^2}\left(1 - \frac{L^2}{rM} + \frac{3L^2}{r^2}\right) \longrightarrow r_\pm = \frac{L^2}{2M}\left(1 \pm \sqrt{L^2 - 12M^2}\right)$$

$L^2 < 12M^2$

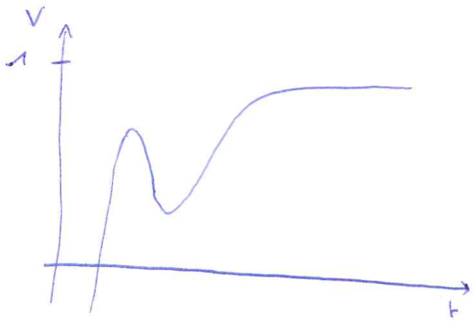


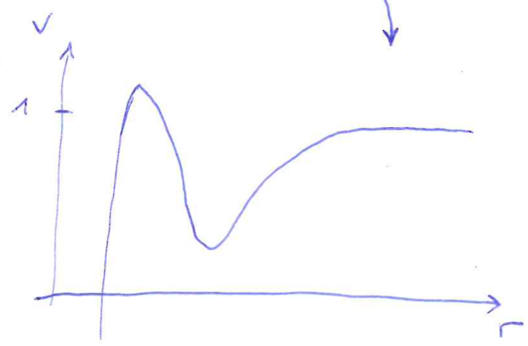
$L^2 > 12M^2$

r_- is a maximum, r_+ is a minimum

$12M^2 < L^2 < 16M^2 \longrightarrow V_{max} < 1$

$L^2 > 16M^2 \longrightarrow V_{max} > 1$





- fall in BH $\forall E$

- $V_{min} < E < V_{max} \longrightarrow$ "elliptic" orbit
- $E > V_{max} \longrightarrow$ fall in BH

- $E^2 > V_{max} \longrightarrow$ fall in BH
- $1 < E^2 < V_{max} \longrightarrow$ turning point in r_0, back to ∞
- $V_{min} < E^2 < 1 \longrightarrow$ bound orbit, approximate ELLIPS

When $L^2 = 12M^2 \longrightarrow r_\pm = 6M \longleftarrow$ smallest possible circular stable orbit

Possible exercises: Radial fall into a BH (spaceship emitting signals to observer at infinity;
deflection of light by the Sun;
shift of the perihelion of Mercury;
Event Horizon Telescope measures)

MOTION AROUND KERR (Some examples!)

Consider an observer falling into the BH with zero angular momentum $L = u_\varphi = 0$

ZAMO $\longrightarrow$ Zero Angular Momentum Observer

$$\Omega = \frac{d\varphi}{dt} = \frac{d\varphi/d\tau}{dt/d\tau} = \underbrace{\frac{u^\varphi}{u^t} = 0}_{r=+\infty} \qquad \text{because} \quad u^\varphi = g^{\varphi\mu}u_\mu = u_\varphi = 0$$

For $r < +\infty$ $\quad \cancel{u^\varphi = g^{\varphi\mu}u_\mu = g^{\varphi t}u_t \neq 0 \to \Omega \neq 0}$ $\quad \cancel{u^t = g^{t\mu}u_\mu = g^{tt}u_t \to \Omega}$

$$u_\varphi = 0 = g_{\varphi\varphi}u^\varphi + g_{t\varphi}u^t$$

$$\Omega = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{2Mar}{(r^2+a^2)^2 - a^2\Delta \sin^2\vartheta}$$

Since $(r^2+a^2)^2 > a^2 \sin^2\vartheta\,(r^2 + a^2 - 2Mr)$ then $\frac{\Omega}{Ma} > 0 \longrightarrow$ ZAMO is co-rotating with the BH