Lecture 1 - The Schwarzschild metric

A bit of conventions ...

- mostly (+) signature (Mateja ~~s~~showed in SR class)
- sometimes I will forget G_N and $c = 1$. You can reinstate them by dimensional analysis

"Who knows how to solve $ax^2 + bx + c = 0$?" $\rightarrow$ Easy equation. EE are not so easy

(EE) $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu}$$

(vacuum EE) $$R_{\mu\nu} = 0$$

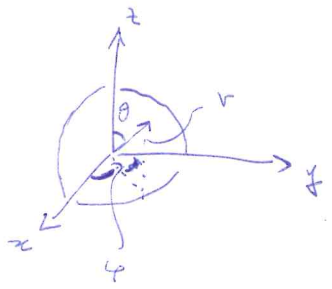
we want BH!

$\rightarrow$
- spherically symmetric BH, simple
- axially symmetric BH, less simple, still analytic
- many BHs moving, hard problem, only numerical

The two cases are simplified by SIMMETRIES

Spherical symmetry: what does it mean? Specify the coordinates

$$x^\mu = (t, r, \theta, \varphi)$$



Minkowsky in flat space

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$$

$$ds^2_M = -dt^2 + dr^2 + r^2 d\Omega^2$$

there can be functions

(this assumption can be more formal in the context of Killing symmetries, more on this in PS2, by Mateja)

Generic static and spherically symmetric metric

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + e^{2\gamma(r)} r^2 d\Omega^2$$

Coordinate redefinition

$$\bar{r} = e^{\gamma(r)} r \quad \rightarrow \quad d\bar{r} = \left(1 + r\frac{d\gamma}{dr}\right) e^{\gamma} dr$$

$$ds^2 = -\underset{\downarrow\, e^{2\alpha(r)}}{e^{2\alpha(r)}} dt^2 + \underbrace{\left(1 + r\frac{d\gamma}{dr}\right)^{-2} e^{2\beta(r) - 2\gamma(r)}}_{e^{2\beta(r)}} d\bar{r}^2 + \underset{\downarrow\, \bar{r}\to r}{\bar{r}^2} d\Omega^2$$

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$$

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta}\right)$$

EXAMPLE

$$\Gamma^t_{tr} = \frac{1}{2} g^{tt} \left(\cancel{\partial_t g_{tr}} + \partial_r g_{tt} - \cancel{\partial_t g_{tr}}\right)$$

$$= \frac{1}{2} e^{-2\alpha} \partial_r e^{2\alpha} = \partial_r \alpha$$

Exercise: Show that the non-zero coefficients are Γ^r_{tt}, Γ^r_{rr}, $\Gamma^\theta_{r\theta}$, $\Gamma^r_{\theta\theta}$, $\Gamma^\varphi_{r\varphi}$, $\Gamma^r_{\varphi\varphi}$, $\Gamma^\theta_{\varphi\varphi}$, $\Gamma^\varphi_{\theta\varphi}$

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} - \Gamma^\alpha{}_{\kappa\nu}\Gamma^\kappa{}_{\beta\mu} + \Gamma^\alpha{}_{\kappa\mu}\Gamma^\kappa{}_{\beta\nu}$$

EXAMPLE $$R^t{}_{rtr} = \cancel{\Gamma^t{}_{rr,t}} - \Gamma^t{}_{rt,r} - \Gamma^t{}_{\kappa r}\Gamma^\kappa{}_{rt} + \Gamma^t{}_{\kappa t}\Gamma^\kappa{}_{rr}$$

$$= -\partial_r^2 \alpha - (\partial_r \alpha)^2 + \partial_r \alpha\, \Gamma^r_{rr}$$

Exercise: Show that the non-zero components are $R^t{}_{\theta t\theta}$, $R^t{}_{\varphi t\varphi}$, $R^r{}_{\theta r\theta}$, $R^r{}_{\varphi r\varphi}$, $R^\theta{}_{\varphi\theta\varphi}$

This is already quite tedious ... People nowadays do it with math manipulation programs, like Wolfram Mathematica

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta} \longrightarrow R_{tt} = e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right]$$

$$R_{rr} = -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta$$

$$R_{\theta\theta} = e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1$$

$$R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta}$$

Take $e^{2(\beta-\alpha)} R_{tt} + R_{rr} = 0$

$$= \frac{2}{r}(\partial_r\alpha + \partial_r\beta) = 0 \longrightarrow \alpha = -\beta + \overset{\text{constant}}{C}$$

The constant can be rescaled away with time reparametrization $t \to e^{-C} t \Longrightarrow \alpha = -\beta$

$$R_{\theta\theta} = 0$$

$$\Rightarrow e^{2\alpha}(2r\partial_r\alpha + 1) = 1 \Longrightarrow \partial_r(re^{2\alpha}) = 1$$

$$re^{2\alpha} = r \mp R_s \longrightarrow e^{2\alpha} = 1 \mp \frac{R_s}{r} \longrightarrow ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

What is the meaning of R_s? One can show that in the weak-field limit

$$g_{tt} \sim -\left(1 + \frac{2\Phi}{c^2}\right)$$ where Φ is the gravitational potential of the Poisson equation $\frac{d^2\vec{x}}{dt^2} = -\nabla\Phi$

For a point particle $\Phi = -\frac{GM}{r} \longrightarrow R_s = \frac{2GM}{c^2}$ Schwarzschild radius!

Static and spherically symmetric metric found! There is more ...

BIRKHHOFF THEOREM

If $\alpha = \alpha(t,r)$ and $\beta = \beta(t,r)$, then

$$\begin{cases} R_{tr} = \frac{2}{r}\frac{\partial\beta}{\partial t} = 0 \longrightarrow \beta = \beta(r) \\ R_{\theta\theta} = e^{-2\beta}\left[r\partial_r(\beta-\alpha) - 1\right] + 1 = 0 \longrightarrow \alpha = \bar{\alpha}(r) + \tilde{\alpha}(t) \end{cases}$$

I can always rescale t such that $\alpha = \alpha(r)$.

For an object in spherical symmetry evolving in time, the external metric is Schwarzschild $\rightarrow$ No emission of GW

What happens for $r=0$ and $r = R_s$? We can calculate

$$R_{\alpha\beta\gamma\mu}R^{\alpha\beta\gamma\mu} = \frac{48G^2M^2}{r^6} \longrightarrow \begin{array}{l} r=0 \text{ is a spacetime singularity} \\ r = R_s \text{ is a coordinate singularity} \end{array}$$

REMOVE IT WITH A CHANGE OF COORDINATES

KERR METRIC (in Boyer-Lindquist coordinates)

$$ds^2_{\text{KERR}} = -dt^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2+a^2)\sin^2\theta\, d\varphi^2 + \frac{2Mr}{\Sigma}(a\sin^2\theta\, d\varphi - dt)^2$$

$$\Delta(r) = r^2 - 2Mr + a^2$$

$$\Sigma(r,\theta) = r^2 + a^2\cos^2\theta$$

- M is the BH mass
- aM is the BH angular momentum
- Stationary (independent of t), not static (not invariant for $t \to -t$)
- Axisymmetric (independent of φ)
- Asymptotically flat, reduces to Schwarzschild for $a \to 0$. What is it for $M \to 0$?
- Curvature invariants are singular for $\Sigma = 0$, regular for $\Delta = 0$

SPACELIKE, TIMELIKE, NULL SURFACES

$\Sigma(x^\mu) = 0 \rightarrow$ hypersurface

$n_\alpha = \Sigma_{,\alpha} \rightarrow$ normal vector

$t^\alpha \rightarrow$ tangent vector $\qquad t^\alpha = \dfrac{dx^\alpha}{d\lambda} \qquad x^\alpha(\lambda)$ curve on Σ

$$t^\alpha n_\alpha = \frac{dx^\alpha}{d\lambda}\frac{\partial \Sigma}{\partial x^\alpha} = \frac{d\Sigma}{d\lambda} = 0$$

In a local inertial frame $\qquad n^\alpha = (n^0, n^1, 0, 0)$

$$n_\alpha n^\alpha = (n^1)^2 - (n^0)^2$$

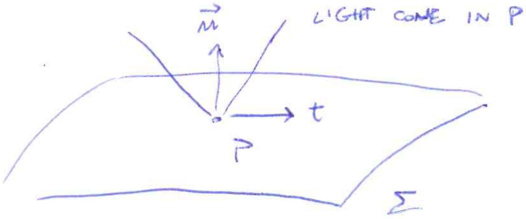
$$n_\alpha t^\alpha = -n^0 t^0 + n^1 t^1 = 0 \quad \rightarrow \quad \frac{t^1}{t^0} = \frac{n^0}{n^1}$$

$$t^\alpha = \lambda(n^1, n^0, a, b) \rightarrow t_\alpha t^\alpha = \lambda^2\left[-n_\alpha n^\alpha + a^2 + b^2\right]$$

- $n_\alpha n^\alpha < 0 \rightarrow t_\alpha t^\alpha > 0$

  SPACELIKE HYPERSURFACE

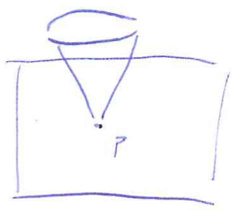
  Can be crossed only in one direction



- $n_\alpha n^\alpha > 0 \rightarrow t_\alpha t^\alpha$ unspecified

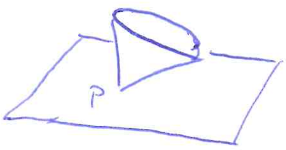
  TIMELIKE HYPERSURFACE

  Can be crossed in any direction



- $n_\alpha n^\alpha = 0 \rightarrow t_\alpha t^\alpha \geq 0$

  NULL HYPERSURFACE



Schwarzschild

$\Sigma \equiv r - \text{const} = 0 \qquad n_\alpha n^\alpha = g^{\alpha\beta}\,\Sigma_{,\alpha}\Sigma_{,\beta} = g^{rr} = \left(1 - \dfrac{2M}{r}\right)$

$r > 2M \rightarrow \Sigma$ is timelike

$r = 2M \rightarrow \Sigma$ is null

$r < 2M \rightarrow \Sigma$ is spacelike