As you may have heard, the main ideas behind GR are

(1) what we perceive as <u>gravity</u>, is really the consequence of <u>space-time</u> being <u>curved</u>.

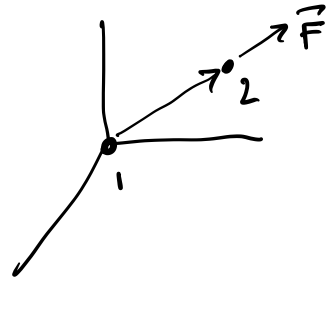
(2) the amount of curvature at each point is controlled by the amount of "stuff" at that point.

The fact that space-time should curve might seem like a very bold leap of faith, that required Einstein's genius. In fact, in these lectures

I will try to argue that, if we try to guess what a relativistic theory of gravity should look like, and we follow our nose, we are led to the conclusion that gravity does look like curved space-time.

Before discussing gravity, let's consider EM as a warmup example. In particular, we will assume as experimental fact the existence of the Coulomb force between two static charges:

$$\vec{F} = \frac{q_1 q_2}{r^2}\,\hat{r}$$



(GAUSSIAN UNITS)

and rederive EM from it demanding Lorentz invariance.

The interesting observation is that $\frac{\vec{F}}{q_2}$ does not depend on any property of the particle 2:

$$\frac{\vec{F}}{q_2} = \frac{q_1}{r^2}\hat{r}$$

this suggests that we can at each point in space we can associate a vector — i.e. we have a <u>vector field</u> — that only depends on the charge of the particle 1 and determines the force / unit charge that any particle next to q_1 will feel. The question we want to answer is: what would this vector field look like if instead of having just one point charge q_1 we had an arbitrarily complicated continuous distribution of charges ρ?

In order to answer this question, we need to figure out what the <u>field equations</u> are — these are the eqs. that allow me to calculate the field if I know ρ. We will <u>not</u> restrict ourselves to the static limit — the regime where Coulomb's law holds — but at the and we will check that Coulomb's law holds in the static limit.

Let's start guessing:

- At this stage nothing looks relativistic. Let's improve that. ρ cannot be the only info we need to specify to calculate the EM field.

In fact, the statement that charge is conserved implies the existence of a current density $\vec{J}$ s.t. the continuity eq. holds:

$$\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0$$

This eq. expresses conservation of charge at any given point.

We can write this in a way that is manifestly Lorentz invariant

$$\partial_\mu J^\mu = 0$$

(<u>Note</u>: Einstein's notation: $\partial_\mu J^\mu \equiv \sum_{\mu=0}^{3} \partial_\mu J^\mu$)

w/ $\partial_\mu \equiv \left(\frac{1}{c} \partial_t , \vec{\nabla} \right)$

$J^\mu \equiv \left(c\rho , \vec{J} \right)$

to make sure that all components have the same dimensions.

[From this, the conservation of <u>total</u> charge of a localized distribution follows by integrating over all space

$$\int d^3r \frac{1}{c} \partial_t J^0 = \frac{d}{dt} \int d^3r \frac{J^0}{c} = \frac{dQ}{dt}$$

$$= -\int d^3r \, \partial_i J^i$$

$$= -\oint_S \vec{J} \cdot \hat{n} \, dS = 0$$

$\vec{J} \to 0$ @ ∞

Under a Lorentz transformation ρ mixes w/ $\vec{J}$:

$$J^\mu \to J^{\mu\prime} = \Lambda^\mu{}_\nu J^\nu$$

This means that our field must be sourced by the 4-vector J^μ

- J^μ should (does) appear w/out derivatives in the eqs., because a configuration $J^\mu = \text{const}$ will generate a non trivial field.

- The relation between J^μ and our field must be linear, because

$$q_1 \to \alpha q_1 \quad \Rightarrow \quad \frac{q_1}{r^2}\hat{r} \to \alpha \frac{q_1}{r^2}\hat{r}$$

- (classical) EM in the static limit should be invariant under the rescaling

$$q \to \lambda q$$
$$\vec{r} \to \lambda \vec{r}$$

Because this doesn't change the Coulomb's law:

$$\vec{F} = \frac{q_1 q_2}{r^2} \hat{r}$$

therefore, this rescaling should leave our field eqs. invariant.

- If we think of our field as a mechanical system, our eqs. must be @ most of second order in time, so that specifying the initial configuration and the rate of change ("velocity") at the initial time is sufficient to determine the field. This means, our eqs must contain @ most 2 ∂_μ's on our field

- We now need to guess which field we want to use

$$\phi \;,\; A_\mu \;,\; B_{\mu\nu} \;,\; C_{\mu\nu\lambda} \;,\; \dots$$

<u>Let's try ϕ</u>. The more general eq. we can write down is $(\Box \equiv \partial_\mu \partial^\mu$

signature $\rightarrow$ $= \partial_t^2 - \nabla^2)$

$$\partial^\mu \phi = J^\mu + \alpha \Box J^\mu + \beta \Box^2 J^\mu$$

normalization absorbed in ϕ (remember $\partial_\mu J^\mu = 0$) $+ \dots$

First, we can redefine

$$\tilde{J}^\mu \equiv J^\mu + \alpha \Box J^\mu + \beta \Box^2 J^\mu + \ldots$$

and we still have $\partial_\mu \tilde{J}^\mu = 0$. $\tilde{J}^\mu$ is as good as J^μ, since the only property we have specified is $\partial_\mu J^\mu = 0$.

Then, our eq. becomes

$$\partial^\mu \phi = \tilde{J}^\mu$$

This is not consistent w/ phenomenology, since not all physical J^μ's can be written as the gradient of a scalar.

<u>Exercise</u>: find one $J^\mu \neq \partial^\mu \chi$

Solution: $J^\mu = (c\, q\, \delta^3(\vec{r}), 0)$

$$\partial_t \chi = q\, \delta^3(\vec{r})$$

$$\rightsquigarrow \chi = q\, t\, \delta^3(\vec{r}) + \chi_0(\vec{r})$$

$$\vec{\nabla} \chi = 0$$

$$\rightsquigarrow q\, t\, \vec{\nabla} \delta^3(\vec{r}) + \vec{\nabla} \phi_0(\vec{r}) \equiv 0$$

cannot be satisfied for all t's. ∎

Main lesson: source + LI places strong constraints on what fields we can consider (convince yourself that the 2-derivative requirement did not play a role in the previous argument that ruled out ϕ)

Let's try A^μ. The most general eq is

$$\partial_\nu \partial^\nu A^\mu + \alpha \partial^\mu \partial_\nu A^\nu + \beta A^\mu$$

$$= \frac{4\pi}{c} J^\mu + \dots$$

LATER CONVENIENCE

can be absorbed in a redefinition of J^μ

We will now fix α and β.

First, we will demand that

$$\left.\begin{aligned} q &\to \lambda q \\ \vec{r} &\to \lambda \vec{r} \end{aligned}\right\} \Rightarrow \rho \to \frac{\rho}{\lambda^2}$$

is a symmetry in the static limit.

Focus on the $\mu = 0$ eq:

$$-\nabla^2 A^0 + \beta A^0 = 4\pi \rho$$

$$\to -\frac{1}{\lambda^2}\nabla^2 A^0 + \beta A^0 = \frac{4\pi}{c}\frac{\rho}{\lambda^2}$$

$$\rightsquigarrow \quad -\nabla^2 A^0 + \lambda \beta A^0 = \frac{4\pi}{c}\rho$$

This eq. is invariant only if $\beta = 0$.

Thus we are left with

$$\partial_\nu \partial^\nu A^\mu + \alpha \partial^\mu \partial_\nu A^\nu = \frac{4\pi}{c} J^\mu$$

Second, we demand that this eq. is consistent w/ $\partial_\mu J^\mu = 0$. Acting w/ ∂_μ on the LHS, we find

$$\partial_\nu \partial^\nu \partial_\mu A^\mu (1+\alpha) = 0$$

$\rightsquigarrow \alpha = -1$

Our eqns therefore are forced to be

$$\boxed{\partial_\nu \left(\partial^\nu A^\mu - \partial^\mu A^\nu \right) = \frac{4\pi}{c} J^\mu}$$

which, as we know, are Maxwell's eqs. written in terms of potentials.

<u>Aside</u>: in terms of $\vec{E}$ and $\vec{B}$, they are

$$\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu \qquad (1)$$

$$\varepsilon^{\mu\nu\lambda\rho} \partial_\nu F_{\lambda\rho} = 0 \qquad (2)$$

$(2) \rightsquigarrow F_{\lambda\rho} = \partial_\lambda A_\rho - \partial_\rho A_\lambda$

We are thus lead to a unique eq. if we try to work w/ A_μ. Of course, it takes more work to get Coulomb's law out of this.

First, though, note that our eq. has a special property that the original general eq. didn't have: gauge invariance:

$$A_\mu \to A_\mu + \partial_\mu F$$

In our approach, this is forced upon us by LI, charge conservation and the form of Coulomb's law.

Derivation of Coulomb's law

To derive Coulomb's law, it is helpful to derive the eom for a relativistic point particle from an action principle.

For a free particle, the eom is

$$m \frac{d U^\mu}{d \lambda} = 0$$

determines shape of worldline

$$\text{w/ } U^\mu = \frac{1}{\sqrt{\frac{dX^\nu}{d\lambda} \frac{dX_\nu}{d\lambda}}} \frac{dX^\mu}{d\lambda}$$

We can derive this eom from the action

$$S_{p.p.} = -mc \int d\lambda \sqrt{\frac{dX^\mu}{d\lambda} \frac{dX_\mu}{d\lambda}}$$

by demanding $\dfrac{\delta S_{p.p.}}{\delta X^\mu} = 0$

<u>Exercise</u>: check this.

When the particle interacts w/ A_μ, then

$$m \frac{d U^\mu}{d\lambda} = \ldots$$

and we want to figure out what "..." is. To this end, notice that Maxwell's eqs. can be derived from an action

$$S = \int d^4x \left\{ -\frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu} \right.$$

$$- \frac{1}{c^2} A_\mu J^\mu \Big\}$$

w/ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

by varying

$$\frac{\delta S}{\delta A_\mu(t, \vec{r})} = 0$$

Exercise : check this

Note : this is just a generalization of the usual variational principle in mechanics : $\dfrac{\delta S}{\delta q_i} = 0$

w/ $i = (\mu, \vec{r})$

For a single point particle

$$J^\mu = \int d\lambda \, c q \frac{dX^\mu}{d\lambda} \delta^4(x - X)$$

This expression is invariant under reparametrizations $\lambda \to \lambda'(\lambda)$.

We can use this to set $X^0(\lambda) = \lambda$ and then:

$$\leadsto J^\mu = \int dX^0 \, c q \left(1, \frac{\vec{v}}{c}\right) \cdot \delta^3(\vec{r} - \vec{X}) \, \delta(ct - X^0)$$

$$= (c \underbrace{q \delta^3(\vec{r} - \vec{X})}_{\rho}, \underbrace{q \vec{v} \delta^3(\vec{r} - \vec{X})}_{\vec{J}})$$

as expected.

Plugging this particular expression in the action and adding $S_{p.p.}$ we get the total action

$$S_{tot} = \int d^4x \left\{ -\frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu} \right\}$$

$$- \int d\lambda \, \frac{q}{c} \, \frac{dX^\mu}{d\lambda} A_\mu(X)$$

$$- mc \int d\lambda \sqrt{\frac{dX^\mu}{d\lambda} \frac{dX_\mu}{d\lambda}}$$

NOTE: relative coeff. between S_A and $S_{p.p.}$ can always be absorbed in m

If we now vary

$$\frac{\delta S}{\delta X^\mu} = 0$$

we get from the 2^{nd} term:

$$-\delta \int d\lambda \, \frac{q}{c} \frac{dX^\mu}{d\lambda} A_\mu(X)$$

$$= -\frac{q}{c} \int d\lambda \left\{ -\delta X^\mu \partial_\nu A_\mu \frac{dX^\nu}{d\lambda} + \frac{dX^\mu}{d\lambda} \partial_\nu A_\mu \delta X^\nu \right\}$$

$$\boxed{= -\frac{q}{c} \int d\lambda \, \delta X^\nu F_{\nu\mu} \frac{dX^\mu}{d\lambda}}$$

and from the 3rd one

$$-mc\,\delta \int d\lambda \sqrt{\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}}$$

$$\boxed{= +mc \int d\lambda \, \delta x^\nu \frac{d}{d\lambda} \left\{ \frac{1}{\sqrt{\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}}} \frac{dx^\nu}{d\lambda} \right\}}$$

pulling it all together we get

$$\boxed{m c^2 \frac{d U^\mu}{d\lambda} = q F^\mu{}_\nu \frac{d x^\nu}{d\lambda}} \qquad (1)$$

together w/

$$\begin{cases} \partial_\nu F^{\nu\mu} = \frac{4\pi}{c} J^\mu \\ F^{\nu\mu} = \partial^\nu A^\mu - \partial^\mu A^\nu \end{cases} \qquad (2)$$

this specifies everything.

We are now ready to derive Coulomb's law.

1. we solve (2) for a static

Charge at the origin:

$$J^\mu = (q_1 c\, \delta^3(\vec{r}), \vec{0})$$

Because J^μ doesn't depend on time, we will look for a solution that is time independent.

Then, using $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$, we have $\rightsquigarrow$ e.g. $\partial^i = -\partial_i$

$$\mu = 0 : \quad -\nabla^2 A^0 = 4\pi q_1 \ \delta^3(\vec{r})$$

$$\mu = i : \quad -\nabla^2 A^i - \partial^i \partial_j A^j = 0$$

Assuming $A^\mu \xrightarrow[r\to\infty]{} 0$, the second eq. implies $\boxed{A^i = 0}$.

The first eq. instead has the solution

$$A^0 = \frac{q_1}{r}$$

Let us now calculate the force on a static particle w mass m_2 and charge q_2 using (1).

Setting $X^0(\lambda) = \lambda$, the $\mu = i$ component is

$$m c^2 \frac{d U^i}{d X^0} = \boxed{m \frac{d^2 \vec{X}}{dt^2}}$$

$$= q_2 \partial^i A_0$$

$$= - q_2 \partial_i A^0$$

$$\boxed{= \frac{q_1 q_2}{r^2} \hat{r}}$$

✓

Here is what we have achieved:

1. assumed
   - Coulomb's law
   - LI
   - conservation of charge
   - 2 derivatives

   to derive Maxwell's eq. and gauge invariance

2. used action principle to derive force law on particles and check consistency w/ Coulomb's law.

Next time we will apply the same strategy to gravity.