The linear eqs. we derived in the previous lecture are very useful and can be used to derive a variety of results (light bending, light retardation, Lense-Thirring effect, GW's, ...)

$\uparrow$ gravitational field of spinning objects is different from static ones.

However, it is important to realize that these eqs. cannot be exact. To see this, we need to move away from the static limit.

Let's first look at how things do work out in E & M. Remember that the covariant eom for a point particle is

$$m c \frac{d U^\mu}{d\tau} = q F^\mu{}_\nu \frac{d x^\nu}{d\tau}$$

Take now the $T_{\mu\nu}$ of the point particle we saw in the previous lecture:

$$T^{\mu\nu} = \int d\lambda \frac{m c^2}{\sqrt{\frac{dX^\rho}{d\lambda}\frac{dX_\rho}{d\lambda}}} \frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}$$

$$\cdot\ \delta^3(\bar{x} - \vec{X}(\lambda))\, \delta(ct - X^0(\lambda))$$

Let us now act on it w/ ∂_μ and then integrate over the entire volume

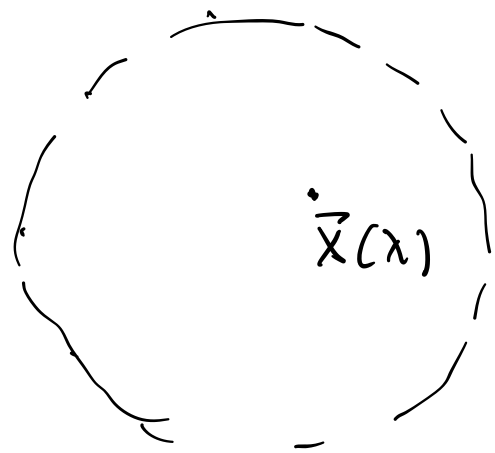
$$\int d^3x\, \partial_\mu T^{\mu\nu} = \int d^3x \left\{ \frac{1}{c}\partial_t T^{0\nu} + \right.$$

$$\left. + \partial_i T^{i\nu} \right\}$$

The second term is the integral of a divergence for any fixed ν:

$$= \oint_{S_{r=\infty}} dS \, \hat{r}_i \, T^{i\nu} = 0$$

$$T^{i\nu} \sim \delta^3(\vec{x} - \vec{X}(\lambda))$$



The first term can be written as

$$\frac{1}{c}\frac{d}{dt}\int d^3x \; T^{0\nu} =$$

$$= \frac{1}{c}\frac{d}{dt}\int d^3x \int d\lambda \, \frac{mc^2}{\sqrt{\frac{dX^\rho}{d\lambda}\frac{dX_\rho}{d\lambda}}} \, \frac{dX^0}{d\lambda}\,\frac{dX^\nu}{d\lambda}$$

$$\cdot \; \delta^3(\vec{x} - \vec{X}(\lambda)) \, \delta(ct - X^0(\lambda))$$

$$= \frac{1}{c}\frac{d}{dt}\int d^3x \int dX^0 \frac{mc^2}{\sqrt{\frac{dX^\rho}{dX^0}\frac{dX_\rho}{dX^0}}}\frac{dX^\nu}{dX^0}$$

$$\delta^3(\vec{x}-\vec{X}(\lambda))\,\delta(ct - X^0(\lambda))$$

$$= \frac{1}{c}\frac{d}{dt}\left\{\frac{mc^2}{\sqrt{\frac{dX^\rho}{dX^0}\frac{dX_\rho}{dX^0}}}\frac{dX^\nu}{dX^0}\Bigg|_{X^0=ct}\right\}$$

$$= \frac{1}{c}\frac{d\tau}{dt}\left\{mc\frac{dU^\nu}{d\tau}\right\}_{X_0=ct}$$

Thus, we see that the $T^{\mu\nu}$ of the point particle must be such that:

$$\partial_\mu T^{\mu\nu} \neq 0$$

otherwise we would be in contradiction w/ the eoms!

Physically, this makes sense. As the particle accelerates under the effect of a EM field it gains momentum energy. However, since E and $\vec{p}$ must be conserved they must come from somewhere: the EM field must also have some energy!

The total energy-momentum tensor of the system is therefore

$$T^{\mu\nu}_{TOT} = T^{\mu\nu}_{P.P.} + T^{\mu\nu}_{EM}$$

and the fact that only the <u>total</u> E and $\vec{p}$ are conserved implies

$$\partial_\mu T^{\mu\nu}_{TOT} = 0$$

and in general $\partial_\mu T^{\mu\nu}_{P.P.} \neq 0$.

Let us now try to guess what $T^{\mu\nu}_{EM}$ should be. We have

$$\begin{array}{ccc}
\displaystyle\int d^3x\, \partial_\mu T^{\mu\nu}_{P.P.} & = & \displaystyle\frac{1}{c}\frac{d\tau}{dt}\left\{ mc^2 \frac{dU^\nu}{d\tau} \right\}_{x^0=ct} \\
\parallel & & \parallel \\
\displaystyle -\int d^3x\, \partial_\mu T^{\mu\nu}_{EM} & & \underbrace{\displaystyle\frac{1}{c}\frac{d\tau}{dt}\, q F^{\nu}{}_{\mu} \frac{dX^\mu}{d\tau}\bigg|_{x^0=ct}}_{*}
\end{array}$$

To figure out what $\partial_\mu T^{\mu\nu}_{EM}$ is, we want to rewrite the RHS as an integral over d^3x

$$* = \int d^3x\, \delta^3(\vec{x} - \vec{X}(t)) \frac{1}{c}\frac{d\tau}{dt}\, q F^{\nu}{}_{\mu} \frac{dX^\mu}{d\tau}\bigg|_{x^0=ct}$$

$$= \int d^3x \int dX^0 \, \delta^3(\vec{x} - \vec{X}) \, \delta(ct - X^0) \, .$$

$$\cdot \, q \, F^{\nu}{}_{\mu} \frac{dX^\mu}{dX^0}$$

$$= \int d^3x \int d\lambda \, \delta^3(\vec{x} - \vec{X}) \, \delta(ct - X^0) \, .$$

$$\cdot \, q \, F^{\nu}{}_{\mu} \frac{dX^\mu}{d\lambda}$$

$$= \int d^3x \, \frac{1}{c} F^{\nu}{}_{\mu} J^\mu$$

Therefore, let's guess that

$$\partial_\mu T^{\mu\nu}_{EM} = -\frac{1}{c} F^{\nu}{}_{\mu} J^\mu$$

$$= - F^{\nu}{}_{\mu} \left\{ \frac{1}{4\pi} \partial_\rho F^{\rho\mu} \right\}$$

Can we guess what $T^{\mu\nu}_{EM}$ should be?

Let's look for something quadratic in F that is symmetric:

$$T^{\mu\nu}_{EM} = \alpha F^{\mu}{}_{\rho} F^{\nu\rho} + \beta \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}$$

$$\rightsquigarrow \partial_\mu T^{\mu\nu}_{EM} = \alpha \, \partial_\mu F^{\mu}{}_{\rho} F^{\nu\rho}$$

$$+ \alpha F_{\mu\rho} \partial^\mu F^{\nu\rho}$$

$$+ \beta \, 2 F_{\rho\sigma} \partial^\nu F^{\rho\sigma}$$

In order to match the previous result notice that because

$$F^{\nu\mu} = \partial^\nu A^\mu - \partial^\mu A^\nu ,$$

we have

$$\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} = 0$$

<u>Exercise</u>: check this.

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then, the second term can be rewritten as:

$$\alpha F_{\mu\rho}\,\partial^\mu F^{\nu\rho} =$$

$$= \alpha F_{\mu\rho}\left\{ -\partial^\nu F^{\rho\mu} - \partial^\rho F^{\mu\nu} \right\}$$

$$= \alpha F_{\rho\sigma}\,\partial^\nu F^{\rho\sigma} + \underbrace{\alpha F_{\mu\rho}\,\partial^\rho F^{\nu\mu}}$$

$$= \alpha F_{\rho\mu}\,\partial^\mu F^{\nu\rho}$$

$$= -\alpha F_{\mu\rho}\,\partial^\mu F^{\nu\rho}$$

same up to a sign

$\rightsquigarrow \; \alpha F_{\mu\rho} \partial^\mu F^{\nu\rho} = \frac{1}{2} \alpha F_{\rho\sigma} \partial^\nu F^{\rho\sigma}$

Combining this w/ the other terms we get

$$\partial_\mu T^{\mu\nu}_{EM} = \alpha \, \partial_\mu F^\mu{}_\rho F^{\nu\rho} + \left(2\beta + \frac{\alpha}{2}\right) F_{\rho\sigma} \partial^\nu F^{\rho\sigma}$$

$$\rightsquigarrow \begin{cases} \alpha = -\dfrac{1}{4\pi} \\ \beta = -\dfrac{\alpha}{4} = +\dfrac{1}{16} \end{cases}$$

$\rightsquigarrow$

$$\boxed{T^{\mu\nu}_{EM} = -\frac{1}{4\pi}\left\{F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4\pi} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}\right\}}$$

Exercise: check that T^{00} is the usual energy density in gaussian units ($T^{00} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$) and T^{0i} is the Poynting vector

$$T^{0i} = \frac{1}{4\pi}(\vec{E} \times \vec{B})^i$$

Hint: use $\quad A^\mu = (\phi, A_x, A_y, A_z)$

and $$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Note: There is a systematic way of obtaining $T^{\mu\nu}$ directly from the action. We won't go into details, but the only observation that is important for our purposes is

$$L \sim (\partial A)^2 \sim T^{\mu\nu}$$

This is true in general.

Recap: we have shown that the point particle eom. is compatible w/ conservation of E and $\vec{P}$ only if the EM field has a $T^{\mu\nu}_{EM}$.

Furthermore,

$$\partial_\mu \left(T^{\mu\nu}_{P.P} + T^{\mu\nu}_{EM} \right) = 0$$

while in general $\partial_\mu T^{\mu\nu}_{P.P} \neq 0$

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Let us now turn to gravity, and

we immediately see that there is a <u>problem</u>. In our derivation of the field eqs, we assumed

$$\partial_\mu T^{\mu\nu}_{p.p} = 0$$

But this is incompatible w/ the eoms we obtained,

$$\frac{d}{d\tau}\left\{ m \frac{dX_\sigma}{d\tau} - \frac{km}{2c^2} h_{\mu\nu} \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\frac{dX_\sigma}{d\tau} \right.$$

$$\left. + \; k m h_{\sigma\nu} \frac{dX^\nu}{d\tau} \right\} =$$

$$= \frac{km}{2} \partial_\sigma h_{\mu\nu} \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}$$

Because of the terms $\sim h_{\mu\nu}$.

We understand why: its because

$$\partial_\mu \left(T^{\mu\nu}_{p.p} + \Delta T^{\mu\nu} \right) = 0$$

$\uparrow$ depends on $h_{\mu\nu}$

On the other hand, our derivation of the eqns relied on the fact that the source $T^{\mu\nu}$ must be conserved.

This suggests that our action should be

$$S = \int d^4x \left\{ (\partial h)^2 - k\, h_{\mu\nu} \left(T^{\mu\nu}_{p.p.} + \Delta T^{\mu\nu} \right) + L_{p.p.} \right\}$$

Remember however that

$$T_{p.p.} + \Delta T \sim L$$

$$\sim (\partial h)^2 + k h \left(T_{p.p} + \Delta T \right) + T_{p.p.}$$

This is a matrix eq. for $\Delta T^{\mu\nu}$

It admits a consistent solution, with a schematic form that is easy to figure out

$$\rightsquigarrow (1 - kh)\,\Delta T \sim (\partial h)^2 + kh\, T_{p.p}$$

$$\rightsquigarrow \quad T_{GRAVITY} \sim \frac{(\partial h)^2}{1-kh} + \frac{kh\,T_{p.p.}}{1-kh}$$

$$\sim (\partial h)^2 + kh\,(\partial h)^2 + k^2h^2\,(\partial h)^2 + \dots$$

$$+ kh\,T_{p.p.} + k^2h^2\,T_{p.p} + \dots$$

$$\boxed{\text{\underline{Note}: unlike in EM, the additional term also has a contribution that depends on } T_{p.p.}}$$

Plugging this back into the Lagrangian,

We see that consistency requires the action to have the form:

$$S = \int d^4x \Big\{ (\partial h)^2 + k\, h\, (\partial h)^2 + k^2 h^2 (\partial h)^2 + k^3 h^3 (\partial h)^2 + \dots + \left(k\, h + k^2 h^2 + k^3 h^3 + \dots\right) T_{p.p} + L_{p.p.} \Big\}$$

If we vary this action, we get

$$\partial^2 h + k h\, \partial^2 h + k^2 h^2 \partial^2 h + \dots = -k\, T \left(k h + k^2 h^2 + \dots \right)$$

$\rightsquigarrow$ $\boxed{\text{unlike for EM, the field eqs. for gravity must be \underline{non-linear}}}$

However,

> the linear eqs provide an accurate description in the weak field regime s.t. $k h \ll 1$

If we can figure out what all the non-linear terms are explicitly, then we can use our eqs to figure out what happens in the strong field regime $k h \gtrsim 1$.

Rather than following the strategy outlined above to derive these terms, we will try to be clever and understand the physical meaning of the point particle action.

The part of the action that depends on the point particle is:

$$S_{p.p.} = -mc^2 \int d\tau - \int d\tau \frac{k}{2} m \, h_{\mu\nu}(X) \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau}$$

$$= -mc \int d\lambda \sqrt{\eta_{\mu\nu} \frac{dX^\mu}{d\lambda} \frac{dX^\nu}{d\lambda}}$$

$$- mc \int d\lambda \frac{1}{2} \frac{k \, h_{\mu\nu}}{\sqrt{\eta_{\rho\sigma} \frac{dX^\rho}{d\lambda} \frac{dX^\sigma}{d\lambda}}} \frac{dX^\mu}{d\lambda} \frac{dX^\nu}{d\lambda}$$

$$\boxed{\simeq -mc \int d\lambda \sqrt{(\eta_{\mu\nu} + k h_{\mu\nu}(X)) \frac{dX^\mu}{d\lambda} \frac{dX^\nu}{d\lambda}} \quad k \, h_{\mu\nu} \ll 1}$$

The last step is a guess for how the last line of

$$S = \int d^4x \Big\{ (\partial h)^2 + k\, h\, (\partial h)^2 + k^2 h^2 (\partial h)^2 + k^3 h^3 (\partial h)^2 + \dots \underbrace{+ \left(k\, h + k^2 h^2 + k^3 h^3 + \dots\right) T_{p.p} + L_{p.p.}}\Big\}$$

should renorm to.

If this guess is correct, then the particle seems to be moving in a spacetime where the Minkowski metric $\eta_{\mu\nu}$ has been modified a bit to become

$$\eta_{\mu\nu} + k\,h_{\mu\nu}.$$

Remember that $\eta_{\mu\nu}$ is what we use to calculate the distance between events in space-time

$$-ds^2 = \eta_{\mu\nu}\, dX^\mu\, dX^\nu$$

What we found is that (at least in the regime where $k h_{\mu\nu} \ll 1$), a point particle behaves as if the distance between space-time points depended on the gravitational field.

What if $k h_{\mu\nu} \gtrsim 1$? It seems natural to assume that know

$$-ds^2 = g_{\mu\nu}(x)\, dX^\mu dX^\nu \quad (*)$$

metric

w/ $g_{\mu\nu}(x)$ in principle very different from $\eta_{\mu\nu}$.

Eq. (*) is very suggestive, because it is how one would calculate the distance between two points on a <u>curved space</u>, where the notion of distance changes from point to point depending on how the space is squeezed or stretched.

have quantitative in the next lecture

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To conclude, let's come back to the gauge symmetry we discovered working w/ the linear theory:

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

This also admits a nice geometric interpretation.

<u>First</u> of all, because this symmetry was derived in the linear approximation, we have no right to assume that it is <u>exact</u>. <u>Second</u>, even in the linear approximation where $k\, h_{\mu\nu} \ll 1$, we now understand that we need

to restrict ourselves to ξ^μ s.t.

$$k\,\partial\xi \ll 1$$

in order for $h'_{\mu\nu}$ to also remain within the regime of validity of the linear approximation, i.e. $k h'_{\mu\nu} \ll 1$.

<u>Third</u>, physical quantities should be invariant under gauge transf. In particular, this applies to the infinitesimal distance between events

$$-ds^{2\,\prime} = (\eta_{\mu\nu} + k h'_{\mu\nu})\, dX^\mu dX^\nu$$

$$= (\eta_{\mu\nu} + k h_{\mu\nu} + k\,\partial_\mu \xi_\nu + k\,\partial_\nu \xi_\mu)\, dX^\mu dX^\nu$$

$$\simeq (\eta_{\mu\nu} + k\, h_{\mu\nu})\, d(x^\mu + k\xi^\mu)\, d(x^\nu + k\xi^\nu)$$

$$\equiv -ds^2 = (\eta_{\mu\nu} + k\, h_{\mu\nu})\, dx^\mu dx^\nu$$

<u>Exercise</u>: here we dropped higher order terms. Check.

This result shows that we can also interpret our gauge invariance as the statement that physics shouldn't change if we transform the coordinates as follows

$$x^\mu \to x'^\mu = x^\mu + \underbrace{k\, \xi^\mu(x)}$$

arbitrary functions s.t. $k\xi^\mu \ll 1$.

What about away from the linear regime? It seems natural to assume that now physics shouldn't depend on arbitrary coordinate transformations i.e. <u>diffeomorphisms</u>:

$$x^\mu \to x'^\mu = f^\mu(x)$$

<u>Bottomline</u>: we are lead to the conclusion that we should work w/ a theory where space-time is curved, the metric $g_{\mu\nu}$ describes gravity, and our theory should

be invariant under diffeomorphisms

The next lecture we will first talk about the geometry of curved spaces, then use this insight to figure out how to rewrite the first two lines of our action.