In this final lecture we will try to be more quantitative about how we describe space-time curvature, and derive Einstein's eqs. of GR.

Remember: we concluded that what we should do is allow for the distance between two space-time points to be given by

$$-ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$$

where $g_{\mu\nu}(x)$ encodes gravity, and we want our theory to be invariant under arbitrary coordinate transformations

$$x^\mu \rightarrow x'^\mu = f^\mu(x) \equiv x'^\mu(x)$$

↳ we demand they are <u>INVERTIBLE</u>, i.e. <u>DIFFS</u>

Note: This field content ($g_{\mu\nu}(x)$) and these symmetries (diffs), together w/ the requirement that field eqs contain @ most 2 derivatives is what *defines* GR.

Any alternative theory of gravity requires changing one or more of these 3 assumptions

First, let's ask ourselves: how does $g_{\mu\nu}(x)$ change if we change coordinates? Since ds^2 is something physical, it cannot depend on which coordinates we choose. Therefore,

$$ds^2 = g'_{\mu\nu}(x')\, dx'^{\mu}\, dx'^{\nu}$$

$$= g_{\mu\nu}(x)\, dx^{\mu}\, dx^{\nu}$$

$$= g_{\mu\nu}(x) \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} dx^{\alpha} dx^{\beta}$$

$$\rightsquigarrow \boxed{g'_{\alpha\beta}(x') = g_{\mu\nu}(x) \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}}}$$

This shows right away that having a

$g_{\mu\nu}(x)$ rather than $\eta_{\mu\nu}$ is not enough to tell that space-time is curved, because we can always start w/ $g_{\mu\nu}(x) = \eta_{\mu\nu}$ and perform a (very complicated) diff to get

$$g'_{\alpha\beta}(x') = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta}$$

This is e.g. what happens if we work in spherical coords:

$$-ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\varphi^2$$

coordinate dependence, but we know that this is just regular Minkowski.

We need a better measure for what it means for a space to be curved.

Before continuing our search, a <u>side remark</u>: we figured out how $g_{\mu\nu}(x)$ changes under diffs. How about other quantities? Take

$$\frac{dx^\mu}{d\lambda} \rightarrow \frac{dx'^\mu}{d\lambda} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{dx^\nu}{d\lambda}$$

<u>note</u>: unlike $g_{\mu\nu}$, primed coordinates are in the numerator.

In fact, this is how all 4-vectors w/ an upper index transform:

$$\boxed{V^{\mu'}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}(x)}$$

Using the metric, we can define a vector w/ a lower index:

$$V_{\mu}(x) \equiv g_{\mu\nu}(x)\, V^{\nu}(x)$$

how does this transform?

$$\boxed{V'_{\mu}(x')} = g'_{\mu\nu}(x')\, V^{\nu'}(x')$$

$$= g_{\alpha\beta}(x) \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \underbrace{\frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x'^{\nu}}{\partial x^{\gamma}}}_{= \frac{\partial x^{\beta}}{\partial x^{\gamma}} = \delta^{\beta}_{\gamma}} V^{\gamma}(x)$$

$$= \frac{\partial x^{\alpha}}{\partial x'^{\mu}} g_{\alpha\beta}(x)\, V^{\beta}(x) = \boxed{\frac{\partial x^{\alpha}}{\partial x'^{\mu}} V_{\alpha}(x)}$$

This generalizes easily to <u>tensors</u> w/ an arbitrary number of up and down indices:

$$\boxed{\begin{aligned} & A^{\mu_1 \ldots \mu_n}{}'_{\nu_1 \ldots \nu_m}(x') = \\ & = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots A^{\alpha_1 \ldots \alpha_n}{}_{\beta_1 \ldots \beta_m}(x) \end{aligned}}$$

<u>Special cases:</u>

NO INDICES: $\phi'(x') = \phi(x)$ (scalar)

E-M TENSOR: $T^{\mu\nu\prime}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} T^{\alpha\beta}(x)$

<u>Exercise</u>: convince yourself that for Poincaré transformations

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + c^{\nu}$$

the rules above reduce to the usual ones.

Let's keep searching, and consider the point particle action w/ $\eta_{\mu\nu} \to g_{\mu\nu}(x)$:

$$S = -mc \int d\lambda \sqrt{g_{\mu\nu}(x) \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda}}$$

If we vary his action w.r.t. x^{μ} we get the p.p. eom, which is of the form

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\nu\rho} \frac{dx^{\nu}}{d\lambda} \frac{dx^{\rho}}{d\lambda} = 0$$

GEODETIC EQ.

CHRISTOFFEL SYMBOL

w/

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_{\nu} g_{\alpha\rho} + \partial_{\rho} g_{\alpha\nu} - \partial_{\alpha} g_{\nu\rho} \right.$$

Could $\Gamma^{\mu}_{\nu\rho}$ be a better way of assessing if the space-time is curved? After all, in Minkowski the p.p. eom is

$$\frac{d^2 x^\mu}{d\lambda^2} = 0$$

and thus one could be tempted to say that if $\Gamma^{\lambda}_{\mu\nu} \neq 0$ then there is a gravitational field. Remember however that this statement needs to be invariant under arbitrary coordinate transf.

<u>Exercise</u>: perform a coordinate transf.

$$x^\mu \longrightarrow x'^\mu = x'^\mu(x)$$

and show that the Christoffel symbols changes as follows:

$$\Gamma'^{\rho}_{\mu\nu}(x') = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x'^{\rho}}{\partial x^{\gamma}} \Gamma^{\gamma}_{\alpha\beta}(x) + \frac{\partial x'^{\rho}}{\partial x^{\gamma}} \frac{\partial^2 x^{\gamma}}{\partial x'^{\mu} \partial x'^{\nu}}$$

$$= \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial x'^{\rho}}{\partial x^{\gamma}} \Gamma^{\gamma}_{\alpha\beta}(x) - \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial^2 x'^{\rho}}{\partial x^{\alpha} \partial x^{\beta}}$$

and show that these two expressions are equivalent.

This means that we could start w/ $\Gamma^{\mu}_{\nu\alpha} = 0$ like in Minkowski, and then perform a diff to end up with $\Gamma'^{\alpha}_{\beta\gamma} \neq 0$.

Therefore, this is not a good criterion.

In fact, demanding that physics does not depend on the coordinates we use is a source of many complications.

For example, what does it mean for a vector field to be constant everywhere? Naively one would think $\partial_\alpha V^\mu(x) = 0$.

However, under a diff we have

$$\partial'_\alpha V^{\mu\prime}(x') = \frac{\partial x^\beta}{\partial x'^\alpha} \partial_\beta \left\{ \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x) \right\}$$

$$= \frac{\partial x^\beta}{\partial x'^\alpha} \left\{ \frac{\partial^2 x'^\mu}{\partial x^\beta \partial x^\nu} V_\nu(x) + \frac{\partial x'^\mu}{\partial x^\nu} \partial_\beta V^\nu(x) \right\}$$

Because of the first term, even if $\partial_\beta V^\nu(x) = 0$, we can end up with $\partial'_\alpha V^{\mu'}(x') \neq 0$.

However, we can fix this by defining a <u>covariant derivative</u> using the Christoffel symbol:

$$\boxed{\nabla_\mu V^\nu(x) \equiv \partial_\mu V^\nu(x) + \Gamma^\nu_{\mu\rho} V^\rho(x)}$$

Exercise: show that

$$\nabla'_\alpha V^{\mu'}(x') = \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial x'^\mu}{\partial x^\nu}\nabla_\beta V^\nu(x)$$

Because $\dfrac{\partial x^\beta}{\partial x'^\alpha}$ are invertible, we have that

$$\nabla'_\alpha V^{\mu'} = 0 \iff \nabla_\beta V^\nu = 0$$

Hence, this provides a good criterion to define whether something is constant or not.

Exercise: show that the covariant derivative of an arbitrary tensor should be

$$\nabla_\alpha T^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m} =$$

$$\partial_\alpha T^{\mu_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m}$$

$$+ \Gamma^{\mu_1}_{\alpha \beta_1} T^{\beta_1 \ldots \mu_n}{}_{\nu_1 \ldots \nu_m} + \ldots$$

$$- \Gamma^{\beta_1}_{\alpha \nu_1} T^{\mu_1 \ldots \mu_n}{}_{\beta_1 \ldots \nu_m} - \ldots$$

Exercise: Show that

$$\nabla_\alpha g_{\mu\nu} = 0$$

this is known as metric compatibility

Let us now take two covariant derivatives of a vector, and something interesting happens: unlike ordinary partial derivatives (for sufficiently smooth fields), covariant derivatives do not commute:

$$[\nabla_\mu, \nabla_\nu] V^\alpha(x) =$$

$$= \nabla_\mu \nabla_\nu V^\alpha - \nabla_\nu \nabla_\mu V^\alpha$$

$$= R^\alpha{}_{\beta\mu\nu} V^\beta$$

↖ RIEMANN TENSOR

w/

$$R^\alpha{}_{\beta\mu\nu} \equiv \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\gamma} \Gamma^\gamma_{\nu\beta} - \Gamma^\alpha_{\nu\gamma} \Gamma^\gamma_{\mu\beta}$$

Based on the transformation properties of covariant derivatives and V^μ, it is easy to see that the Riemann tensor transforms as

$$R'^{\alpha}{}_{\beta\mu\nu}(x') = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} \frac{\partial x^{\lambda}}{\partial x'^{\mu}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} \cdot R^{\gamma}{}_{\delta\lambda\rho}$$

Therefore, if $R^{\alpha}{}_{\beta\mu\nu} \neq 0$ in some coordinates, it will be so in all coordinates.

$$\boxed{\text{The Riemann tensor, and therefore whether or not covariant derivatives commute, is a meaningful measure of the curvature of space-time.}}$$

Some properties of the Riemann tensor:

1. $R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$
2. $R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$
3. $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$
4. $\epsilon^{\mu\nu\lambda\rho} R_{\mu\nu\lambda\rho} = 0$

ALGEBRAIC

ϵ: LEVI CIVITA TENSOR (totally antisymmetric)

5. $\nabla_\rho R_{\mu\nu\alpha\beta} + \nabla_\mu R_{\nu\rho\alpha\beta} + \nabla_\nu R_{\rho\mu\alpha\beta} = 0$

DIFFERENTIAL (Bianchi identities)

Exercise: argue that $R_{\mu\nu\alpha\beta}$ has only 20 independent components

Solution: First use 1.–3.

$$R_{\underbrace{\alpha\beta}_{I}\underbrace{\mu\nu}_{J}}$$

w/ I, J taking 6 values each, and symmetry under $I \leftrightarrow J$

a symmetric $N \times N$ matrix has

$\frac{N(N+1)}{2}$ independent components

$N = 6 \rightsquigarrow \frac{6 \times 7}{2} = 21$

4. is one more constraint,

$\rightsquigarrow 21 - 1 = \boxed{20}$

We can define two other useful quantities:

<u>Ricci Tensor</u>: $R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}$

<u>Ricci scalar</u>: $R = g^{\mu\nu} R_{\mu\nu}$

<u>Note</u>: $R_{\mu\nu}$ contains just 10 combinations of the 20 independent quantities. Therefore, $R_{\mu\nu}$ could

be zero even if the space-time is curved, i.e. $R^{\alpha}{}_{\beta\mu\nu} \neq 0$.

This happens e.g. for black holes.

After this long detour on geometry, let's come back to physics. Remember that we are trying to guess what the LHS of the eqs

$$\partial^2 h + k h \partial^2 h + k^2 h^2 \partial^2 h$$

$$+ \dots = -k T^{FULL}$$

should resum to. (For a point-particle, we have already guessed an action that should yield the RHS.)

We have suppressed indices, but the

LHS should carry two indices, be symmetric, and have two derivatives. These are the same properties of $R_{\mu\nu}$. Furthermore, when $g_{\mu\nu} = \eta_{\mu\nu} + k h_{\mu\nu}$, $R_{\mu\nu}$ contains an infinite number of powers of $h_{\mu\nu}$.

<u>Exercise</u>: convince yourself that $R_{\mu\nu}$ contains an ∞ number of powers of $h_{\mu\nu}$

The leading linear term in $R_{\mu\nu}$ is

$$R_{\mu\nu} = -\frac{k}{2}\left\{ \Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \left(\partial_\sigma \partial_\nu h^\sigma{}_\mu + \partial_\sigma \partial_\mu h^\sigma{}_\nu \right) \right\}$$

which is not quite the term in our linear eqs:

$$\Box h_{\mu\nu} + \partial_\mu \partial_\nu h$$

$$- \left(\partial_\lambda \partial_\mu h_\nu{}^\lambda + \partial_\lambda \partial_\mu h_\nu{}^\lambda \right)$$

$$- \eta_{\mu\nu} \Box h + \eta_{\mu\nu} \partial_\lambda \partial_\rho h^{\lambda\rho}$$

Because we are missing the last line, however,

$$R = g^{\lambda\rho} R_{\lambda\rho} \simeq k \left(\partial_\lambda \partial_\rho h^{\lambda\rho} - \Box h \right)$$

and therefore the combination

$$- \frac{2}{k} \left\{ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right\}$$

reproduces exactly the correct linear order.

Therefore, we guess that our eqs should be

$$-\frac{2}{k}\left\{R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right\}=-k\,T_{\mu\nu}^{\text{FULL}}$$

and using $k=\sqrt{\frac{16\pi G}{c^3}}$

$$\boxed{\rightsquigarrow\ R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi G\,T_{\mu\nu}^{\text{FULL}}\qquad (c\equiv 1)}$$

EINSTEIN'S EQUATIONS

<u>Note about conservation</u>: contract the Bianchi identity twice, and we find

$$\nabla_\mu\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)=0$$

$$\rightsquigarrow \nabla_\mu T^{\mu\nu}_{\text{FULL}} = \partial_\mu T^{\mu\nu}_{\text{FULL}} + \Gamma^{\mu}_{\mu\rho} T^{\rho\nu}_{\text{FULL}} + \Gamma^{\nu}_{\mu\rho} T^{\mu\rho}_{\text{FULL}} = 0$$

contain $h_{\mu\nu}$

Thus, $T^{\mu\nu}_{\text{FULL}}$ is covariantly conserved

while $T^{\mu\nu} + \Delta T^{\mu\nu}$ is regularly conserved, i.e. $\supset T^{\mu\nu}(\kappa h)^n,\ \partial^2(\kappa h)^n$

$$\partial_\mu \left(T^{\mu\nu} + \Delta T^{\mu\nu}\right) = 0$$

Note about action: we managed to renorm our eqs. Can we also renorm the action? Remember, the part that depends only on $h_{\mu\nu}$ looks like

$$S = \int d^4x \left\{ (\partial h)^2 + k h (\partial h)^2 + k^2 h^2 (\partial h)^2 + \cdots \right\}$$

what we know is that we should be able to write this only in terms of $g_{\mu\nu}$, w/ at most two derivatives, and that this action should be invariant under diffs. One quantity that is invariant under diffs would be R, but

$$\int d^4x \, R$$

would not be invariant because d^4x is not.

$$d^4x \to d^4x' = d^4x \det \frac{\partial x'}{\partial x}$$

but this is easy to fix, because

$$\det g_{\mu\nu}(x) \rightarrow \det g'_{\mu\nu}(x')$$

$$= \det \left\{ g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \right\}$$

$$= \det g_{\alpha\beta} \cdot \left(\det \frac{\partial x}{\partial x'} \right)^2$$

$$= \det g_{\alpha\beta} \left(\det \frac{\partial x'}{\partial x} \right)^{-2}$$

$$\leadsto \quad d^4 x' \sqrt{-\det g'_{\alpha\beta}(x')}$$

$$= d^4 x \sqrt{-\det g_{\alpha\beta}(x)}$$

$$\leadsto \quad \boxed{S = \int d^4 x \sqrt{-\det g} \; \frac{R}{k^2}}$$

We are now in a position to see

that this is actually not the only action we could have written that is invariant and has @ most 2 derivatives. More in general, we could have

$$\int d^n x \sqrt{-\det g}\ \frac{1}{k^2}(R - 2\Lambda)$$

cosmological constant

This term curves space-time even in the absence of other stuff, and Minkowski is no longer a solution to Einstein's eqs w/ $T_{\mu\nu}=0$, which now became

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = 0}$$