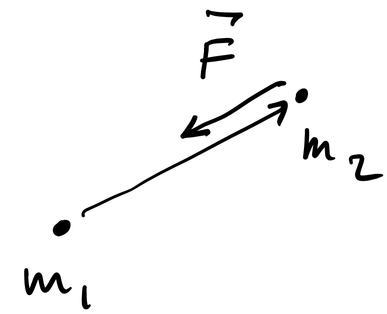
We are now going to play the same game we played in the first lecture, but w/ gravity. Our goal now is to find a <u>relativistic field theory</u> that reproduces the Newton force between two particles at rest:

$$\vec{F} = -\frac{G m_1 m_2}{r^2}\hat{r}$$

NOTE!



The first we need to do is to <u>identify the source</u> in our field equation.

What plays the role of J^μ?

If there is one thing we know about special relativity, is that $E = mc^2$ for particles @ rest. This suggests that energy should be sourcing our gravitational field. But this cannot be the end of the story. As we know, for a point particle $P^\mu = (E, \vec{P})$, i.e. energy and momentum mix under Lorentz transformations:

$$P^\mu \to P^{\mu'} = \Lambda^\mu{}_\nu P^\nu$$

Therefore, if energy sources gravity, so does <u>momentum</u>. These will be the analog of the charge Q. Remember that, because charge is conserved, the density of charge ρ satisfied an eq

$$\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0$$

and therefore ρ was the $\mu = 0$ component of a 4-vector J^μ.

Now, P^μ are 4 conserved quantities. This means that their densities must be the $\mu = 0$ component of a tensor,

$$\rho^\mu \equiv T^{\mu 0}$$

which satisfies the following conservation equation:

$$\partial_\nu T^{\mu\nu} = 0$$

$T^{\mu\nu}$ is the <u>energy</u>-<u>momentum</u> (or <u>stress-energy</u>) tensor, and will be our relativistic source (the analog of J^μ)

Physical meaning of components:

T^{00} = energy density

T^{0i} = energy current

T^{i0} = momentum density

T^{ij} = momentum current

Interestingly, not all of these components are physically independent. Conservation of energy and momentum follow from invariance under translations. Invariance under Lorentz transformations places further constraints on $T^{\mu\nu}$. It we break it up into symmetric and anti-symmetric part,

$$T^{\mu\nu} \equiv T_S^{\mu\nu} + T_A^{\mu\nu}$$

w/ $$T_S^{\mu\nu} = T_S^{\nu\mu}$$

$$T_A^{\mu\nu} = -T_A^{\nu\mu}$$

then it can be shown that Lorentz invariance implies

$$T_A^{\mu\nu} = \partial_\lambda X^{\lambda\mu\nu}$$

w/ totally antisymmetric.

This means that

$$\partial_\mu T_A^{\mu\nu} = \partial_\mu \partial_\lambda X^{\lambda\mu\nu} = 0$$

and therefore

$$\partial_\mu T_S^{\mu\nu} = 0$$

i.e., in a Lorentz invariant theory we can always choose $T^{\mu\nu}$ to be symmetric. This implies, in particular,

$$\boxed{\text{energy current} = \text{momentum density}}$$

<u>Proof</u> (to be omitted during the lecture)

The canonical energy-momentum tensor is:

$$\Theta^{\mu\nu} = \Pi^\mu{}_A \, \partial^\nu \phi^A - \eta^{\mu\nu} \mathcal{L}$$

w/ $\Pi^\mu{}_A \equiv \dfrac{\partial \mathcal{L}}{\partial\, \partial_\mu \phi^A}$

The currents associated w/ Lorentz invariance are

$$M^{\mu\lambda\rho} = x^\lambda \Theta^{\mu\rho} - x^\rho \Theta^{\mu\lambda} + \Pi^\mu{}_A \left(\Sigma^{\lambda\rho}\right)^A{}_B \phi^B$$

Lorentz generators in the appropriate representation.

$$\partial_\mu M^{\mu\lambda\rho} = 0 \iff \Theta^{\lambda\rho} - \Theta^{\rho\lambda} = -\partial_\mu \underbrace{\left(\Pi^\mu \cdot \Sigma^{\lambda\rho} \cdot \phi\right)}_{\equiv H^{\mu\lambda\rho}}$$

$$\leadsto \Theta^{\lambda\rho} = S^{\lambda\rho} - \frac{1}{2}\partial_\mu H^{\mu\lambda\rho}$$

SYMMETRIC

$$\equiv T_S^{\lambda\rho} - \frac{1}{2}\,\partial_\mu\left\{H^{\mu\lambda\rho} - H^{\lambda\mu\rho} - H^{\rho\mu\lambda}\right\}$$

where we have just extracted a symmetric term from $S^{\lambda\rho}$, which means that $T_S^{\lambda\rho}$ is still symmetric. This gives us an explicit form for $X^{\mu\lambda\rho}$:

$$X^{\mu\lambda\rho} = -\frac{1}{2}\,\partial_\mu\left\{H^{\mu\lambda\rho} - H^{\lambda\mu\rho} - H^{\rho\mu\lambda}\right\}$$

<u>Note</u>: If we impose additionally conformal symmetry, then the trace of $T_S^{\mu\nu}$ must also be a total derivative:

$$T_S{}^\mu{}_\mu = \partial_\mu Y^\mu$$

Let's recap: we have concluded that our source should be the energy-momentum tensor $T^{\mu\nu}$, and that $T^{\mu\nu} = T^{\nu\mu}$.

Once again, because $\vec{F} \sim m_1$, we will look for field eqs. that are linear, and @ most second order.

In the case of EM we saw that the EM field A^μ was a vector field like the source J^μ (i.e., same representation of the Lorentz group).

Thus, it seems like a good guess to try working w/ a field $h_{\mu\nu}$ that is symmetric. Let's write down the most general linear eqs. w/ @ most two derivatives:

$$
\left[\begin{array}{l} \Box \equiv \partial_\mu \partial^\mu \\ h \equiv h_\sigma{}^\sigma \\ T \equiv T_\sigma{}^\sigma \end{array}\right]
$$

$$
\begin{aligned}
&\Box h^{\mu\nu} + a\, \partial^\mu \partial^\nu h \\
&+ b\left(\partial_\lambda \partial^\nu h^{\mu\lambda} + \partial_\lambda \partial^\mu h^{\nu\lambda} \right) \\
&+ c\, \eta^{\mu\nu} \Box h \quad + d\, \eta^{\mu\nu} \partial_\lambda \partial_\rho h^{\lambda\rho} \\
&+ e\, h^{\mu\nu} + f\, \eta^{\mu\nu} h \quad = -k T^{\mu\nu} \\
&\underbrace{+ \xi\, \eta^{\mu\nu} T}_{1} \quad \underbrace{+ \alpha \Box T^{\mu\nu} + \beta\, \partial^\mu \partial^\nu T + \gamma\, \eta^{\mu\nu} \Box T + \ldots}_{2}
\end{aligned}
$$

1 Take the trace of this eq to get

$$
\ldots = \underbrace{-kT + 4\xi T}_{= (4\xi - k) T} + \ldots
$$

Multiply by $\dfrac{\xi}{4\xi - k}\, \eta^{\mu\nu}$ both sides and subtract this eq from the first one to get

$$\square h^{\mu\nu} + a' \partial^\mu \partial^\nu h$$

$$+ b' \left(\partial_\lambda \partial^\nu h^{\mu\lambda} + \partial_\lambda \partial^\mu h^{\nu\lambda} \right)$$

$$+ c' \eta^{\mu\nu} \square h + d' \eta^{\mu\nu} \partial_\lambda \partial_\rho h^{\lambda\rho}$$

$$+ e' h^{\mu\nu} + f' \eta^{\mu\nu} h$$

$$= - k' T^{\mu\nu} + \alpha' \square T^{\mu\nu} + \beta' \partial^\mu \partial^\nu T + \gamma' \eta^{\mu\nu} \square T + \dots$$

Thus we can effectively set $\boxed{\xi = 0}$.

Exercise: check this and relate the new parameters to the old ones.

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(2) we can rewrite the eqns schematically as

$$-k' T^{\mu\nu} = \partial^2 h + \partial^2 T + \cdots$$

$$= \partial^2 h + \partial^2(\partial^2 h + \partial^2 T + \cdots) + \cdots$$

$$= \partial^2 h + \partial^4 h + \partial^4 T + \cdots$$

$$= \partial^2 h + \partial^4 h + \partial^6 h + \cdots$$

Since we don't want more than 2 derivatives, we will set $\boxed{\alpha = \beta = \gamma = \cdots = 0}$ and work with (dropping the primes)

$$\Box h^{\mu\nu} + a\, \partial^\mu \partial_\nu h$$

$$+ b\left(\partial_\lambda \partial^\nu h^{\mu\lambda} + \partial_\lambda \partial^\mu h^{\nu\lambda}\right)$$

$$+ c\, \eta^{\mu\nu} \Box h + d\, \eta^{\mu\nu} \partial_\lambda \partial_\rho h^{\lambda\rho}$$

$$+ e\, h^{\mu\nu} + f\, \eta^{\mu\nu} h = -k T^{\mu\nu}$$

Once again, we can use the classical symmetry of Newton's law,

$$m_i \to \lambda m_i$$

$$r \to \lambda r$$

To infer that the eoms should be invariant under

$$T^{\mu\nu} \sim \frac{m}{r^3} \longrightarrow \frac{T^{\mu\nu}}{\lambda^2}$$

$$x^\mu \to \lambda x^\mu$$

This symmetry implies $\boxed{e = f = 0}$

Therefore we are left with

$$\Box h^{\mu\nu} + a\, \partial^\mu \partial_\nu h$$

$$+ b\left(\partial_\lambda \partial^\nu h^{\mu\lambda} + \partial_\lambda \partial^\mu h^{\nu\lambda} \right)$$

$$+ c\,\eta^{\mu\nu}\,\Box h + d\,\eta^{\mu\nu}\,\partial_\lambda \partial_\rho h^{\lambda\rho}$$

$$= -k\,T^{\mu\nu}$$

Let us now demand consistency w/ $\partial_\mu T^{\mu\nu} = 0$.
Acting w/ ∂_μ on the LHS we find:

$$\underline{\Box\,\partial_\mu h^{\mu\nu}} + \underline{a\,\Box\,\partial^\nu h}$$

$$+ b\left(\partial^\nu \partial_\lambda \partial_\mu h^{\mu\lambda} + \underline{\Box\,\partial_\lambda h^{\lambda\nu}}\right)$$

$$+ \underline{c\,\partial^\nu \Box h} + d\,\partial^\nu \partial_\lambda \partial_\rho h^{\lambda\rho} =$$

$$= (1+b)\,\Box\,\partial_\mu h^{\mu\nu} + (a+c)\,\partial^\nu \Box h$$

$$+ (b+d)\,\partial^\nu \partial_\lambda \partial_\rho h^{\lambda\rho} \equiv 0$$

$$\rightsquigarrow \boxed{b = -1\,, \quad d = -b = 1\,, \quad c = -a}$$

$$\rightsquigarrow \Box h^{\mu\nu} + a\,\partial^\mu\partial^\nu h - \left(\partial_\lambda\partial^\nu h^{\mu\lambda} + \partial_\lambda\partial^\mu h^{\nu\lambda}\right) - a\,\eta^{\mu\nu}\Box h + \eta^{\mu\nu}\partial_\lambda\partial_\rho h^{\lambda\rho} = -k\,T^{\mu\nu}$$

Finally, notice that the value of "a" is not physical, because we can always introduce a new field $\bar{h}^{\mu\nu}$ related to $h^{\mu\nu}$ by

$$h^{\mu\nu} = \bar{h}^{\mu\nu} - c\,\eta^{\mu\nu}\bar{h} \qquad (*)$$

<u>Exercise</u>: invert this relation and solve for $\bar{h}^{\mu\nu}$

Substituting (*) into our eq. we

find an eq. of exactly the same form for $\bar{h}^{\mu\nu}$ w/

$$a \longrightarrow \bar{a} \equiv a(1-4C) + 2C$$

<u>Exercise</u>: check this.

Therefore let's choose C so that $\bar{a} = 1$

$$\rightsquigarrow \boxed{C = \frac{1-a}{2-4a}}$$

<u>Note</u>: $a = 1/2$ doesn't allow us to do this, but would be problematic for other reasons. $\boxed{\text{TBD}}$

<u>interesting</u>: for $a = 1/2$ we have the gauge invariance $h^{\mu\nu} \rightarrow h^{\mu\nu} + \eta^{\mu\nu} f$

We have therefore found that if we want to use a symmetric rank-2 field $h^{\mu\nu}$, its eq. is completely nailed.

$$\boxed{\begin{aligned}&\Box h^{\mu\nu} + \partial^\mu \partial^\nu h\\ &- \left(\partial_\lambda \partial^\nu h^{\mu\lambda} + \partial_\lambda \partial^\mu h^{\nu\lambda}\right)\\ &- \eta^{\mu\nu} \Box h + \eta^{\mu\nu} \partial_\lambda \partial_\rho h^{\lambda\rho} = -k T^{\mu\nu}\end{aligned}}$$

Interestingly, this eq. has a <u>gauge symmetry</u>, like Maxwell's eqs:

$$h^{\mu\nu} \to h^{\mu\nu} + \partial^\mu f^\nu + \partial^\nu f^\mu$$

<u>Exercise</u>: check that.

We now want to check if these eqs.

reproduce Newton's law in the static limit. We know the drill:

1. find an action that reproduces our field eqs.

2. write $T^{\mu\nu}$ for point particle

3. vary this action w.r.t. X^μ to get eoms and take static limit ($\vec{v}=0$)

4. solve static field eqs. w/ $T^{\mu\nu}$ of another point particle.

Let's start. First, we can derive our eqs. from the following action:

$$S = \int d^4x \left\{ \frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} \right.$$

$$- \partial_\lambda h_{\mu\nu} \partial^\nu h^{\mu\lambda} - \partial_\lambda h \, \partial_\rho h^{\lambda\rho}$$

$$+ \frac{1}{2} \partial_\lambda h \, \partial^\lambda h - k \, h_{\mu\nu} T^{\mu\nu} \Big\}$$

<u>Exercise</u>: check this.

Second, I will now argue that the $T^{\mu\nu}$ of a point particle is

$$T^{\mu\nu}(x) = \int d\tau \, m \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} \delta^4(x - X(\lambda))$$

$$\left(c \, d\tau = \sqrt{\frac{dX^\rho}{d\lambda} \frac{dX_\rho}{d\lambda}} \, d\lambda \right)$$

choose again $X^0(\lambda) = \lambda$:

$$\rightsquigarrow T^{\mu\nu} = \int dX^0 \frac{mc}{\sqrt{\frac{dX^\rho}{dX^0}\frac{dX_\rho}{dX^0}}} \frac{dX^\mu}{dX^0}\frac{dX^\nu}{dX^0} \,,$$

$$\cdot\, \delta^3(\vec{x} - \vec{X}(X^0))\, \delta(ct - X^0)$$

$$= \left(\begin{array}{c|c} \dfrac{mc}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} & \dfrac{m\vec{v}}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} \\ \hline \dfrac{m\vec{v}}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} & \dfrac{m v^i v^j/c}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} \end{array}\right) \cdot\, \delta^3(\vec{x} - \vec{X}(t))$$

As expected,

$$T^{0\mu} = \overbrace{\left(\frac{E}{c}, \vec{p}\right)}^{p^\mu} \delta^3(\vec{x} - \vec{X}(t))$$

Plugging our $T^{\mu\nu}$ in the action we find

$$\boxed{\begin{aligned} S = \int d^4x \Big\{ & \frac{1}{4} \partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} \\ & - \frac{1}{2} \partial_\lambda h_{\mu\nu} \partial^\nu h^{\mu\lambda} - \frac{1}{2} \partial_\lambda h \partial_\rho h^{\lambda\rho} \\ & + \frac{1}{4} \partial_\lambda h \partial^\lambda h \Big\} \\ - \int d\tau \, \frac{k}{2} \, & m \, h_{\mu\nu}(X) \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} - mc^2 \int d\tau \end{aligned}}$$

(Relative coeff chosen for later convenience

Let us now vary this action w.r.t. X^μ to get the eoms. We find:

$$\frac{d}{d\tau} \Big\{ m \frac{dX_\sigma}{d\tau} - \frac{km}{2} h_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} \frac{dX_\sigma}{d\tau}$$

$$+ kmc \, h_{\sigma\nu} \frac{dX^\nu}{d\tau} \Big\} =$$

$$= \frac{kmc}{2} \partial_\sigma h_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau}$$

<u>Exercise</u>. Show it.

<u>Solution</u>:

$$\delta S = \delta \int d\lambda \left\{ -\frac{kmc}{2\sqrt{\frac{dX^\rho}{d\lambda}\frac{dX_\rho}{d\lambda}}} h_{\mu\nu} \frac{dX^\mu}{d\lambda} \frac{dX^\nu}{d\lambda} \right.$$

$$\left. - mc \sqrt{\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}} \right\}$$

$$= \int d\lambda \; \delta X^\sigma \left\{ -\frac{d}{d\lambda}\left[\frac{kmc}{2(\quad)^{3/2}} \frac{dX_\sigma}{d\lambda} h_{\mu\nu} \frac{dX^\mu}{d\lambda} \frac{dX^\nu}{d\lambda} \right] \right.$$

$$- \frac{kmc}{2\sqrt{\frac{dX^\rho}{d\lambda}\frac{dX_\rho}{d\lambda}}} \partial_\sigma h_{\mu\nu} \frac{dX^\mu}{d\lambda} \frac{dX^\nu}{d\lambda}$$

$$+ \frac{d}{d\lambda}\left[\frac{kmc}{\sqrt{\frac{dx^\rho}{d\lambda}\frac{dx_\rho}{d\lambda}}} h_{\sigma\nu} \frac{dx^\nu}{d\lambda} \right]$$

$$+ mc \frac{d}{d\lambda}\left[\frac{1}{\sqrt{\frac{dx^\mu}{d\lambda}\frac{dx_\mu}{d\lambda}}} \frac{dx_\sigma}{d\lambda} \right] \Big\} = 0$$

multiplying by $\frac{c}{\sqrt{\frac{dx_\rho}{d\lambda}\frac{dx^\rho}{d\lambda}}}$ we get

$$\boxed{\begin{aligned} &\frac{d}{d\tau}\left\{ m \frac{dX_\sigma}{d\tau} - \frac{km}{2c^2} h_{\mu\nu} \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\frac{dX_\sigma}{d\tau} \right. \\ &\left. + \; km\, h_{\sigma\nu} \frac{dX^\nu}{d\tau} \right\} = \\ &\qquad = \frac{km}{2} \partial_\sigma h_{\mu\nu} \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} \end{aligned}}$$

Let us now consider the static limit ($\vec{v} \to 0$) and take the $\sigma = i$ component:

$$\boxed{m \frac{d^2 x^i}{dt^2} + k m \, h_{ij} \frac{d^2 x^j}{dt^2} = \frac{k m c^2}{2} \partial_i h_{00}}$$

Now, we want to solve our field eqs for the $h_{\mu\nu}$ sourced by another point particle. The eqs look messy, but we can simplify them imposing the Hilbert gauge condition:

$$\partial_\mu \left(h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h \right) = 0$$

Exercise: show that we can always impose this condition.

Then our eqs. become

$$\Box \left(h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h \right) = - k T^{\mu\nu}$$

Thus, if define

$$\phi^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h$$

we have reduced our eqs to

$$\begin{cases} \Box \phi^{\mu\nu} = - k T^{\mu\nu} \\ \partial_\mu \phi^{\mu\nu} = 0 \end{cases}$$

Now, a static point particle @ $\vec{r} = 0$ has

$$T^{\mu\nu} \to \delta^\mu_0 \delta^\nu_0 m_2 c \, \delta^3(\vec{r})$$

and the static eqs become

$$\nabla^2 \phi^{\mu\nu} = \delta^\mu_0 \delta^\nu_0 \, k \, m_2 c \, \delta^3(\vec{r})$$

$$\partial_i \phi^{i\mu} = 0$$

w/ b.c.'s s.t $\phi^{\mu\nu} \to 0$ @ $\vec{r} \to \infty$,

our solutions are

$$\phi^{00} = -\frac{k \, m_2 c}{4\pi r} \qquad \phi^{ij} = 0$$

What about $h^{\mu\nu}$?

<u>Exercise</u>: invert the definition of $\phi^{\mu\nu}$ to find $h^{\mu\nu} = \phi^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\phi$

Thus,

$$h^{00} = \phi^{00} - \frac{1}{2}\phi^{00} = -\frac{k m_2 c}{8\pi r}$$

$$h^{ij} = +\frac{1}{2}\delta^{ij}\phi^{00} = -\frac{k m_2 c}{8\pi r}\delta^{ij}$$

Therefore, our eqns are

$$m\frac{d^2x^i}{dt^2} + \frac{k^2 m m_2 c}{8\pi r}\frac{d^2x^i}{dt^2} = $$

$$= +\frac{k^2 m m_2 c^3}{16\pi}\frac{\hat{r}_i}{r^2}$$

$$= -\frac{k^2 m m_2 c^3 \hat{r}^i}{16\pi r}$$

In order for the RHS to match Newton's law we need

$$\frac{k^2c^3}{16\pi} \equiv G \rightsquigarrow \boxed{k = \sqrt{\frac{16\pi G}{c^3}}}$$

Note however that we have an extra term on the LHS. This is negligible compared to $m\vec{a}$ if :

$$\boxed{\frac{k^2 m_2 c}{8\pi r} = \frac{2Gm_2}{rc^2} \ll 1}$$

Recap: we have found that a field $h_{\mu\nu}$ sourced by $T_{\mu\nu}$ via linear eqs. can reproduce Newton's law, but unlike Coulomb's law, Newton's law is only approximate.

Next lecture: we will see that even our linear field eqs. are only approximate.