Black holes I: GR and Schwarzschild basics

sábado, 27 de julio de 2024 10:55

- **General Relativity**: a geometric theory of relativistic gravitation

Basic object:

Metric $ds^2 = g_{ij}(x)\, dx^i dx^j$: shape and size of spacetime manifold

$i,j = 0,1,2,3$ $\qquad - + + +$

Locally can always set $g_{ij}(x_0^k + \Delta x^k) = \eta_{ij} + O(\Delta x)^2$

η_{ij}: Minkowski metric $(-1,1,1,1)$

$$ds^2 = \eta_{ij}\, dx^i dx^j + O(\Delta x^2)$$
$$= -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + O(\Delta x)^2$$

Can eliminate $O(\Delta x)$ by going to local inertial frame (Equivalence principle)

Coord Trafos: $x^i \to x^{i'}(x^j)$ change the metric. (diffeos)

Infinitesimal: $x^i \to x^i + \varepsilon v^i(x)$ $\qquad \varepsilon \ll 1$

$$g_{ij} \longrightarrow g_{ij} + \varepsilon(\nabla_i v_j + \nabla_j v_i) \qquad \partial_i \equiv \frac{\partial}{\partial x^i}$$

but physical effects must not change under diff.

$$g_{ij}(x_0 + \Delta x) = \eta_{ij} - \frac{1}{3} R_{ikjl}(x_0)\, \Delta x^k \Delta x^l + O(\Delta x)^3$$

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Curvature Tensor (Riemann) $\sim \partial^2 g$

[usual definition in terms of holonomy]

Ricci $R_{ij} = R^k{}_{ikj}$

Scalar curvature $R = R^i{}_i$

Einstein equations:

$$\underbrace{R_{ij} - \frac{1}{2} g_{ij} R}_{\text{Geometry}} = 8\pi \frac{G}{c^2} \underbrace{T_{ij}}_{\text{Matter}}$$

Relativistic Gravity } G and c allow to connect (transform) matter and geometry

$$\text{Curvature} = \frac{G}{c^2} \text{ Mass density}$$

Characteristic size of object $\sim R$

Curvature radius $\sim R_c$

$$\frac{1}{R_c^2} \sim \frac{G}{c^2}\frac{M}{R^3} \equiv \frac{L_M}{R^3} \qquad L_M \equiv \frac{GM}{c^2}$$

(up to O(1) factor)

Mass can be measured in meters

$$1\,\text{Kg} \sim 10^{-29}\,\text{m}$$

Your weight $\sim 10^{-27}$ m $<<$ your physical size

$$L_M << R$$

$$l_M << R$$

$$R_c^2 = R\left(\frac{R}{l_M}\right)^2 >> R \quad \text{low curvature}$$

In order for $R_c \sim R$ high curvature

it must be that $R \sim l_M$

$$R \sim \frac{GM}{c^2} \; : \; \text{black hole}$$

Gravity gravitates on itself

Vacuum equations : $R_{ij} - \frac{1}{2} g_{ij} R = 0$

$$\Leftrightarrow \quad R_{ij} = 0$$

These have non-trivial non-linear solutions

• The Schwarzschild solution

Simplest non-trivial solution :

$$ds^2 = -\left(1 - \frac{r_0}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{r_0}{r}} + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$$(c = 1)$$

- Indep of t : static
- Spherically symmetric
- Length parameter r_0 :

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Newtonian limit: $-g_{tt} \simeq 1 + 2\Phi_N$, $|\Phi_N| \ll 1$

Newtonian potential of localized mass source M:

$$\Phi_N = -\frac{GM}{r} \quad \Rightarrow \quad r_0 = 2GM$$

Restoring c: $r_0 = 2\frac{G}{c^2}M$

as anticipated above

A purely geometric characterization of mass:

$$M = -\frac{1}{8\pi G}\int_{S^2} d^2x\,(K - K_0)$$

(Hawking + Horowitz)

S^2: sphere at large radius

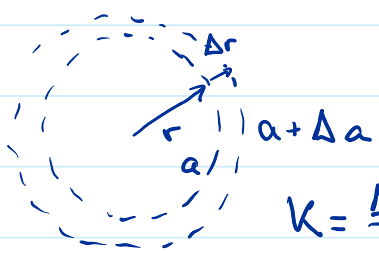
K: Trace of extrinsic curvature of S^2 (mean curvature) in g_{ij}

K_0: " " " " in flat Minkowski background

$$K = \frac{1}{a}\partial_n a$$

a: area element of S^2

∂_n: derivative in unit normal direction



$$K = \frac{\Delta a}{a}$$

In Schw and in Minkowski $K \neq K_0$

Mass $\propto K_0 - K$

In Minkowski: $ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2$ at $r = R$ $t = \text{const}$

$$a = r^2\omega_2 \qquad \omega_2 = \sin\theta \qquad \int d\theta\, d\phi\, \omega_2 = 4\pi$$

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$\partial_n = n^i \partial_i = \partial_r$ $\qquad K_0 = \frac{1}{r^2}\partial_r r^2 = \frac{2}{r}$

In Schwarzschild $ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2$ $\qquad f = 1 - \frac{r_0}{r}$

$a = r^2$

but now $n^r = \sqrt{f}$ so that $n^r n^r g_{rr} = 1$

$\partial_n = n^r \partial_r = \sqrt{f}\,\partial_r$ $\qquad \sqrt{f} \simeq 1 - \frac{r_0}{2r} + \dots$ for $r \gg r_0$

$$K = \frac{1}{r^2}\left(1 - \frac{r_0}{2r}\right)\partial_r r^2 \Big|_{r \gg GM} = \frac{2}{r} - \frac{r_0}{r^2} + \dots$$

$$M = -\frac{1}{8\pi G}\int_{S^2} \sqrt{h}\,(K - K_0) = -\frac{1}{8\pi G}\int d\theta\, d\phi\, \sin\theta\, r^2 \left(\frac{2}{r} - \frac{r_0}{r^2} + \dots - \frac{2}{r}\right)$$

$$= -\frac{1}{8\pi G} 4\pi r^2 \left(-\frac{r_0}{r^2} + \dots\right)_{r\to\infty} = \frac{r_0}{2G} \Rightarrow r_0 = 2GM \checkmark$$

• The Horizon

$(G = c = 1)$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2$$

Metric is singular, as a matrix, at $r = 2M$

But metric changes w/ diffeos.

(non-singular metric: g_{ij} finite, $\det(g_{ij})$ finite (invertible))

Let's shine light:

Let's shine light:

ingoing radial light rays:

θ, ϕ constant

$ds^2 = 0 \quad \Rightarrow \quad dt^2 = \dfrac{dr^2}{\left(1-\frac{2M}{r}\right)^2}$

$$dt = -\frac{dr}{\left|1-\frac{2M}{r}\right|} \qquad : \text{ingoing } \begin{matrix} dt>0 \\ dr<0 \end{matrix}$$

$\Rightarrow \quad t = -r_* + \text{constant}$

where $\quad dr_* = \dfrac{dr}{|1-2M/r|}$

$$r_* = r + 2M \log|r-2M| \qquad (\text{"Tortoise coord"})$$

Note: $\quad r_* \to r \to \infty \quad$ as $r \to \infty$

$\qquad\quad r_* \to -\infty \quad$ as $r \to 2M$

Ingoing light rays are

$$v \equiv t + r_* = \text{constant}$$

As $r \to 2M$ along a light ray, $t \to +\infty$

But t is a coordinate whose meaning is clear near ∞, less so at smaller r.

We may choose v as coordinate, adapted to ingoing light rays, instead of t.

i.e. change $\quad (t, r) \longrightarrow (v, r)$

$$dt = dv - dr_* = dv - \frac{dr}{1-\frac{2M}{r}}$$

$$ds^2 = -\left(1-\frac{2M}{r}\right) dv^2 + 2\,dv\,dr + r^2\, d\Omega_2$$

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$$\left[(g_{ij}) : \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & r^2\Omega_2 \end{pmatrix} \right] : \underline{\text{non-singular at } r=2M}$$

$g_{vv} = 1-\frac{2M}{r}$: finite

$\det g_{ij} = -r^2 \sin^2\theta$
finite

Ingoing light rays encounter $r=2M$ as a smooth place.

Analyze all radial light rays:

$ds^2 = 0 \Rightarrow \left(1-\frac{2M}{r}\right)dv^2 = 2\,dv\,dr$

$\Rightarrow$ $v = \text{const}$: ingoing

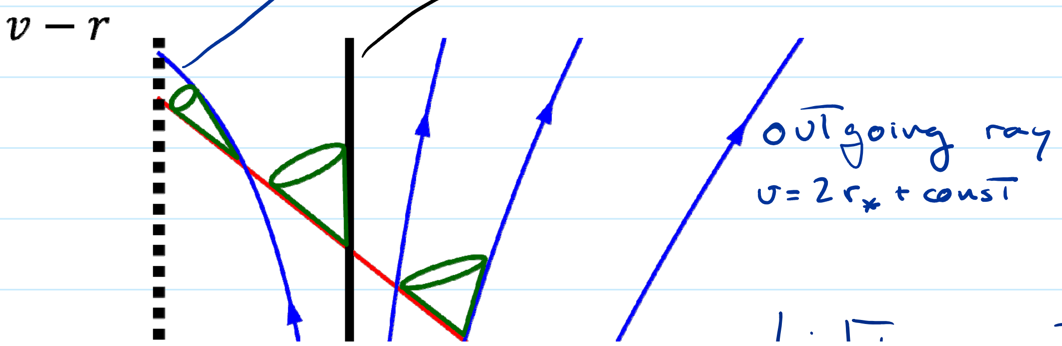
$r = 2M$: constant radius

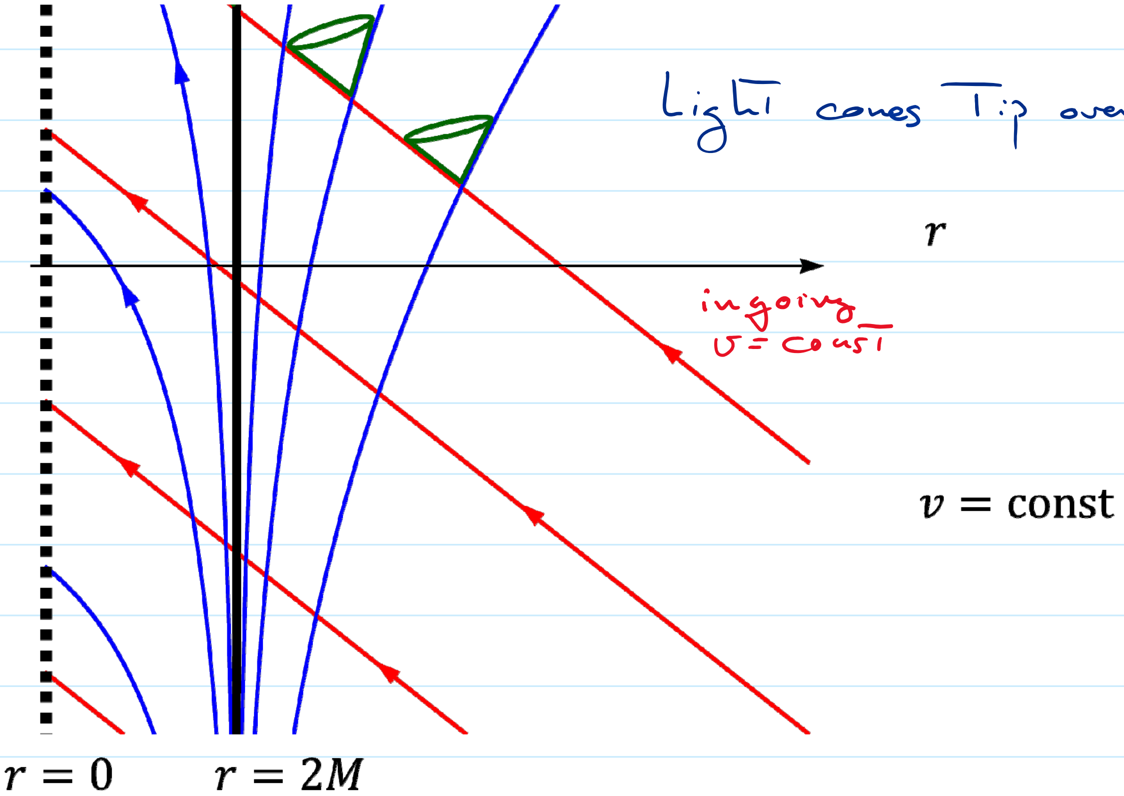
$\left(1-\frac{2M}{r}\right)dv = 2dr \Rightarrow v = 2r_* + \text{const}$: outgoing

Plot $\quad v - r = t + (r_* - r) \sim t$ for $r \gg M$

"outgoing ray" does not escape

"outgoing ray" stays at fixed position





Since we (massive objects) move inside light cones,
then inside $r<2M$ we cannot escape to $r>2M$

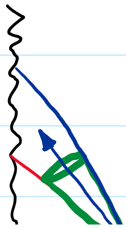
- $\underline{r=2M}$: "frozen light rays" : $\boxed{\text{HORIZON}}$

  [one per point in the sphere:
  "horizon is generated by null rays"]

  Null hypersurface $\quad S^2 \times \mathbb{R}$

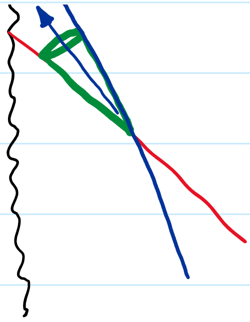
  Separates light rays that can or cannot escape

- $\underline{r=0}$ is a $\boxed{\text{singularity}}$ $\left(R_{ijkl} R^{ijkl} \to \infty \text{ at } r=0\right)$



No light rays emanate from the singularity: it cannot be seen!

It is experienced as an event



It is experienced as an event in the future.

Singularity is not a point in space, but a moment in time

All particle trajectories in $r < 2M$ end at $r = 0$ in finite proper time.

Black holes II: Rindler & Kruskal

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Going near the horizon:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2$$

Close to $r = 2M$ $\quad r = 2M + \xi^2/8M \quad |\xi| << M$

Then

$$ds^2 \simeq \underbrace{-\frac{\xi^2}{16M^2} dt^2 + d\xi^2}_{} + \underbrace{4M^2 d\Omega_2}_{}$$

put aside: large sphere $\sim \mathbb{R}^2$

Similar to $ds^2 = \underbrace{\rho^2 d\phi^2 + d\rho^2}_{\mathbb{R}^2 \text{ in polars}}$ but with $-$ sign

coord singularity at $\rho = 0$ removed

by $x = \rho\cos\phi \quad y = \rho\sin\phi$

Make $\quad X = \xi \cosh t/4M \qquad X^2 - T^2 = \xi^2$

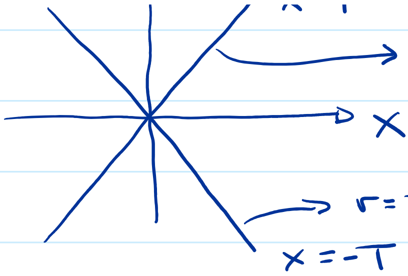
$\quad T = \xi \sinh t/4M \qquad \frac{T}{X} = \tanh t/4M$

Then

$$ds^2 \simeq \underbrace{-dT^2 + dX^2}_{\text{Minkowski}_{1+1}} + \left(dY^2 + dz^2\right)$$

$r = 2M \Leftrightarrow \xi^2 = 0 \Leftrightarrow |X| = |T|$: Metric regular at $r = 2M$

$\rightarrow r = 2M$: null : future horizon H^+



$r=2M$: null : future horizon H^+

$r=2M$: null past horizon H^-

In general, near a (non-degenerate) horizon

$$ds^2 \simeq -\kappa^2 x^2 dt^2 + dx^2 + r_0^2 d\Omega^2$$

κ = surface gravity $\qquad \kappa = \frac{1}{4M}$ for Schw

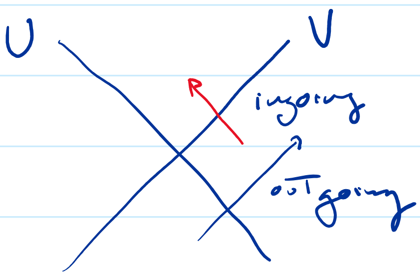
Null coordinates

$v = t + r_*$ $\qquad\qquad$ $r_* = r + 2M \log|r-2M|$

$u = t - r_*$ (ingoing) $\qquad\qquad \simeq 4M \log \xi + \text{const}$

$$V = T + X = \xi e^{t/4M} \simeq e^{\frac{t+r_*}{4M}} = e^{v/4M}$$

$$U = T - X = -\xi e^{-t/4M} \simeq -e^{\frac{-t+r_*}{4M}} = -e^{-u/4M}$$



Can we extend this further away from horizon?

Yes, it works

$$V = 2M e^{v/4M} \qquad U = -2M e^{-u/4M}$$

$$\Downarrow$$

$$2M(r-2M)e^{r/2M} = -T^2 + X^2$$

$$\tanh \frac{t}{4M} = \frac{T}{X}$$

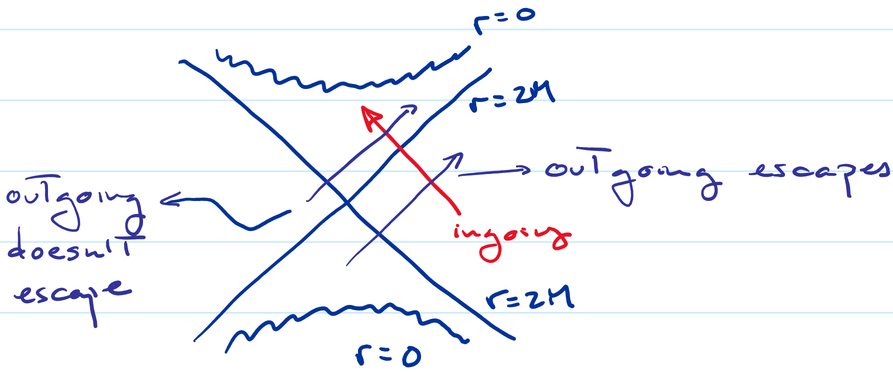
$$ds^2 = 8M \frac{e^{-r/2M}}{r}\left(-dT^2 + dX^2\right) + r^2 d\Omega_2$$ Kruskal

$r(T,X)$ implicit above

No singularity at $r=2M$, only at $r=0$

$r=0 \Leftrightarrow T^2 - X^2 = -4M^2$: hyperbolas

<u>spacelike</u> singularity



Interior <u>is dynamical</u> : T- dependent

(no Timelike Killing vector)

Black holes III: Conformal diagrams

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Causal diagrams (Penrose)

What's the structure of infinity in an asymptotically flat/AdS/dS space?

Is it the same if we go to infinity along timelike trajectories, spacelike, null?

Must assign points at ∞ while preserving causal relationships.

For a spacetime $\mathcal{M}$ with metric $ds^2 = g_{ij}(x)\, dx^i dx^j$

introduce an auxiliary spacetime $\tilde{\mathcal{M}}$ with

$$d\tilde{s}^2 = \Omega^{-2}(x)\, ds^2 \qquad \tilde{g}_{ij} = \Omega^{-2} g_{ij}$$

such that $\Omega^2(x) \to \infty$ as $|x| \to \infty$

and $\tilde{g}_{ij} \to$ finite (more precisely $\Omega \sim |x|$) for non-singular ∞

$\tilde{\mathcal{M}}$ has the same causal structure as $\mathcal{M}$

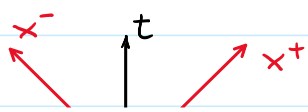
$$ds^2 \lesseqgtr 0 \iff d\tilde{s}^2 \lesseqgtr 0$$

but $\tilde{\mathcal{M}}$ is compact: include points at boundary

$\partial\tilde{\mathcal{M}}$: conformal boundary of $\mathcal{M}$

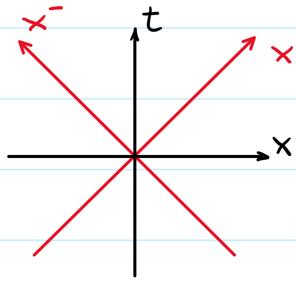
Minkowski$_{1+1}$: $ds^2 = -dt^2 + dx^2 \qquad x^\pm = t \pm x$

$= -dx^+ dx^-$



Minkowski$_{1+1}$: $ds^2 = -dt^2 + dx^2 \qquad x^\pm = t \pm x$

$$= -dx^+ dx^-$$



Bring infty to finite distance through a conformal transformation

$x^+ = \tan u \qquad x^\pm \to \pm\infty \qquad u, v \to -\pi/2 , \pi/2$

$x^- = \tan v$

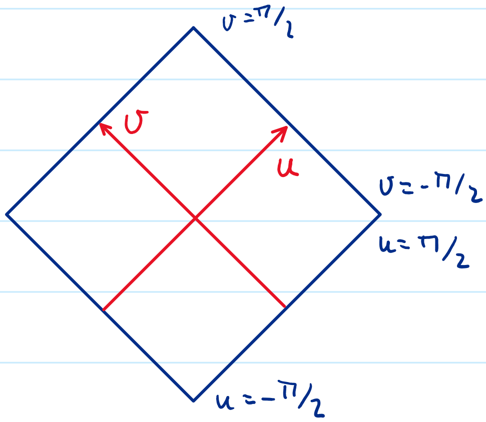
finite range $\quad -\frac{\pi}{2} < u, v < \pi/2$

$$ds^2 = -dx^+ dx^- = -\frac{1}{\cos^2 u \cos^2 v}\, du\, dv$$

Define

$$d\tilde{s}^2 = \Omega^{-2} ds^2 \quad w/ \quad \Omega = \frac{1}{\cos u \cos v}$$

$$= -du\, dv$$



Infinities:

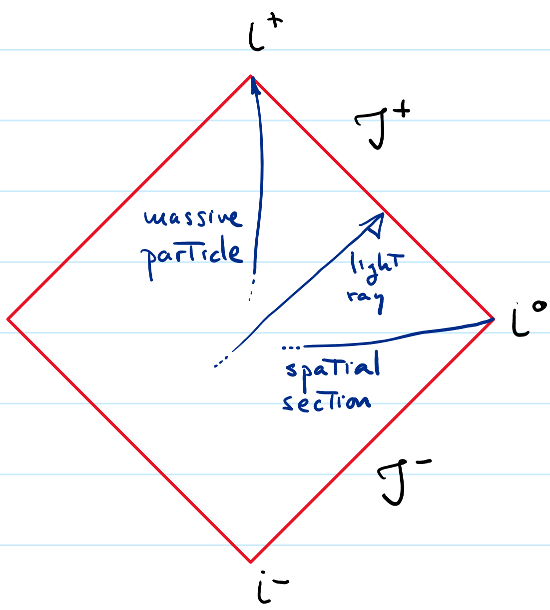
$i^+ = \{ t \to \infty, \text{ fixed } x \}$: future timelike ∞

$i^- = \{ t \to -\infty, \quad \text{"} \quad \text{"} \}$: past timelike ∞

$i^0 = \{ x \to \infty, \text{ fixed } t \}$: spacelike ∞

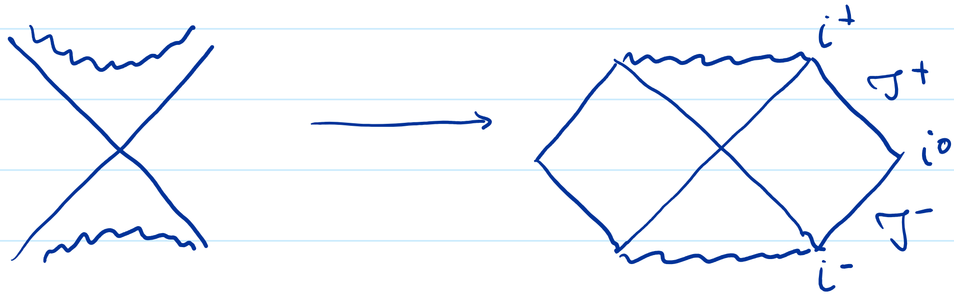
$\mathcal{J}^+ = \{ x^+ \to \infty, \text{ fixed } x^- \}$: future null ∞

$\mathcal{J}^- = \{ x^- \to -\infty, \text{ fixed } x^+ \}$: past null ∞



Asymptotically flat spaces have this same structure near infinity.

Schwarzschild:



Anti de Sitter$_{1+1}$

$$ds^2 = -\left(\frac{r^2}{L^2}+1\right)dt^2 + \frac{dr^2}{\frac{r^2}{L^2}+1}$$

$$= -\cosh^2\frac{\rho}{L}\,dt^2 + d\rho^2$$

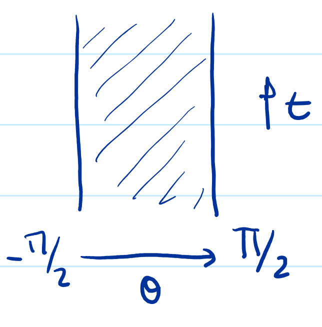
$$\sinh\frac{\rho}{L} = \frac{r}{L} = \tan\theta$$

$$\cosh\frac{\rho}{L} = \frac{1}{\cos\theta}$$

$$= \frac{1}{\cos^2\theta}\left(-dt^2 + L^2 d\theta^2\right)$$

$-\pi/2 \leq \theta \leq \pi/2$

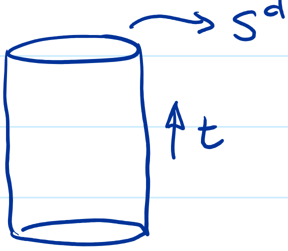
Spatial interval of length L



AdS_{d+1}

$$ds^2 = -\left(\frac{r^2}{L^2}+1\right)dt^2 + \frac{dr^2}{r^2/L^2+1} + r^2 d\Omega_{d-1}$$

$$= \frac{1}{\cos^2\theta}\left(-dt^2 + L^2\underbrace{\left(d\theta^2 + \sin^2\theta\, d\Omega_{d-1}\right)}_{S^d}\right)$$

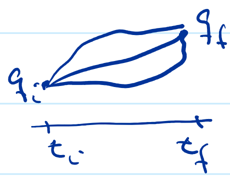


Einstein-Hilbert action

We want to derive the Einstein eqs from a Lagrangian, or from an action, by extremizing it for variations of the metric, $g_{\mu\nu} \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$ that remain fixed at the boundary $\delta g_{\mu\nu}(x)\big|_{\partial M} = 0$

In Classical Mech $q(t) \to q(t) + \delta q(t)$

with $q(t_i)$, $q(t_f)$ fixed



Dirichlet bc's.

These allow to have conserved energy and a well-defined variational problem

$$I = \int dt \, \tfrac{1}{2} \dot{q}(t)^2 \longrightarrow \frac{\delta I}{\delta q(t)} = 0 \Rightarrow \ddot{q} = 0$$

For gravity there's a simple candidate:

$$I_G^{(0)} = \frac{1}{16\pi G} \int_M \sqrt{-g}\, R \qquad \text{Einstein-Hilbert}$$

Vary $g^{ij} \to g^{ij} + \delta g^{ij}$

$$\left[\begin{array}{l} \delta\sqrt{-g} = -\tfrac{1}{2} g_{ij}\, \delta g^{ij} \sqrt{-g} \\ \delta(R_{ij} g^{ij}) = R_{ij}\, \delta g^{ij} + g^{ij}\, \delta R_{ij} \end{array} \right]$$

$$\delta I_G^{(0)} = \frac{1}{16\pi G} \int_M \sqrt{-g} \left(R_{ij} - \tfrac{1}{2} g_{ij} R \right) \delta g^{ij} - \frac{1}{8\pi G} \int_{\partial M} \sqrt{h}\, \delta K$$

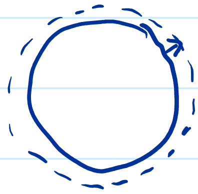
$\big[\, K =$ mean extrinsic curvature of boundary $\partial M \,\big]$

K = extrinsic curvature of boundary ∂M

h_{ij} = metric induced at boundary

n_i = unit normal

$$h_{ij} = g_{ij} \mp n_i n_j$$



$$K = \frac{1}{\sqrt{h}} n^i \partial_i \sqrt{h} = \frac{1}{\sqrt{h}} \partial_n \sqrt{h}$$

relative variation of boundary area along normal

Define

$$I_g = I_g^{(0)} + \frac{1}{8\pi G} \int_{\partial M} \sqrt{h}\, K$$

$\hookrightarrow$ York-Gibbons-Hawking Term (GHY)

so that

$$\frac{\delta I_g}{\delta g^{ij}} = \frac{1}{16\pi G} \left(R_{ij} - \frac{1}{2} g_{ij} R \right)$$

With matter $I_m(\phi, g)$

$$T_{ij} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta I_m}{\delta g^{ij}}$$

$$\frac{\delta (I_g + I_m)}{\delta g^{ij}} = \frac{1}{16\pi G} \left(R_{ij} - \frac{1}{2} g_{ij} R - 8\pi G T_{ij} \right) = 0$$

$\Rightarrow$ Einstein eqns

$\Rightarrow$ Einstein eqs

For a vacuum solution g_{cl}, $R_{ij}(g_{cl})=0$ $\quad R(g_{cl})=0$

$$I_G[g_{cl}] = \frac{1}{8\pi G}\int_{\partial M}\sqrt{h}\,K$$

In asymp flat or AdS this is infinite because of infinite volume. It requires regularization of infinity and subtraction of divergences: renormalization of classical action. In AdS this is holographic renormalization.

Black holes V: Charged black holes

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Charged black holes

$$R_{ij} - \frac{1}{2} g_{ij} R = 8\pi G T_{ij}^{EM}$$

$$\nabla_i F^{ij} = 0 \qquad T_{ij}^{EM} = F_{ik} F_j{}^k - \frac{1}{4} F^2$$

$$I = I_G - \frac{1}{4} \int \sqrt{-g}\, F^2 \qquad F^2 = F_{ij} F^{ij}$$

What can we expect in the metric?

The electromagnetic field energy will gravitate

Say $A_t \sim \frac{Q}{r}$

$$F_{tr} \sim \frac{Q}{r^2} \qquad T_{ij} \sim \frac{Q^2}{r^4}$$

since $\partial^2 g \sim T$

we expect from EM a term $\sim \frac{Q^2}{r^2}$

in the metric

Indeed, solving the Einstein-Maxwell eqns gives

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + Q^2/r^2} + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$A_t = -\frac{Q}{r} \qquad \text{Reissner-Nordström solution}$$

Mass $= M$ from $-g_{tt} = -1 + \frac{2M}{r} + \dots$

Charge $= Q$ from $A_t = -\frac{Q}{r}$ and Gauss' law

$$\left[\begin{aligned} \text{Charge} &= \frac{1}{4\pi} \int_{S^2} d\vec{s} \cdot \vec{E} \qquad \text{with } F^{tr} = \frac{Q}{r^2} = E^r \\ &= \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, r^2 \frac{Q}{r^2} = Q \end{aligned} \right]$$

<u>Singularity</u>: $r=0$ is a curvature singularity: $R_{\mu\nu} R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \xrightarrow{r \to 0} \infty$

[It is a timelike singularity whenever $Q \neq 0$, as we will see.]

<u>Horizons</u>: look for zeroes of g_{tt} (only if static!): $V = 0 \iff \underline{r^2 - 2Mr + Q^2 = 0}$

Horizons : look for zeroes of $\overset{V}{g_{tt}}$: $V=0 \iff r^2 - 2Mr + Q^2 = 0$

$$r = r_\pm = M \pm \sqrt{M^2 - Q^2}$$

Always assume $M > 0$ (negative mass is unphysical)

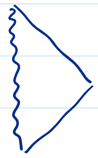
w.l.o.g. $Q \geq 0$ (in the metric only Q^2 appears)

We'll discuss separately the three cases $M < Q$, $M > Q$, $M = Q$

• $M < Q$: no horizon at real r.

$r = 0$: naked singularity (Timelike)

$r=0$



• $M > Q$: There are metric singularities at $r = r_\pm$

Eddington-Finkelstein coordinates

$$dr_* = \frac{dr}{V} \qquad r_* = \int \frac{dr}{V}$$

$$v = t + r_* \qquad\qquad (v, r, \theta, \phi)$$

$$ds^2 = -V dv^2 + 2\,dr\,dv + r^2 d\Omega \quad : \quad \text{regular at } r = r_\pm$$

r_+ : outer horizon

r_- : inner horizon

$$\kappa_\pm = \pm \frac{\sqrt{M^2 - Q^2}}{r_\pm^2}$$

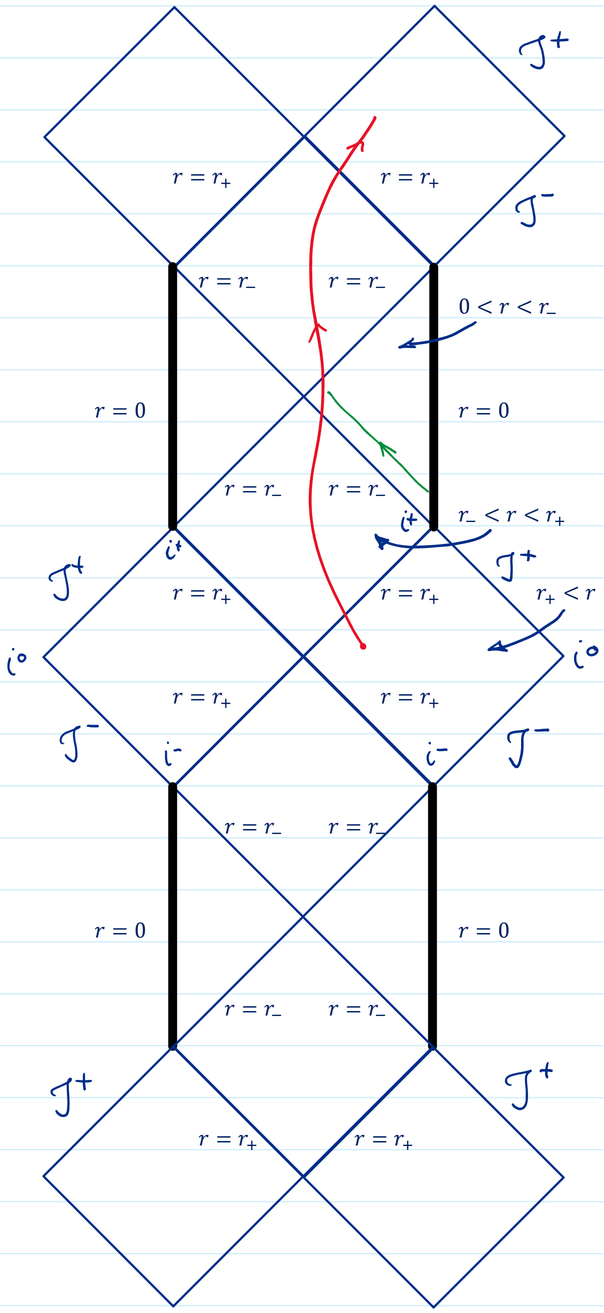
$\kappa_- < 0$ indicates repulsion : one has accelerate to get to it

$r > r_+$: AF region

$r_- < r < r_+$: r Timelike, t spacelike

$0 < r < r_-$: r spacelike, t Timelike

$r = 0$: Timelike singularity

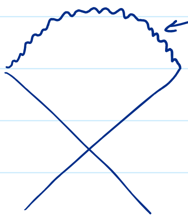


starts in one AF region and travels to a different one, emerging through a white hole

Past r_-, the singularity at $r=0$ is visible

We expect the inner horizon, to be unstable to perturbations, and possibly then the geometry cannot be extended past that singularity.

~~~ ← singularity. It may have portions that are null or spacelike
~~~



← singularity. It may have portions that are null or spacelike

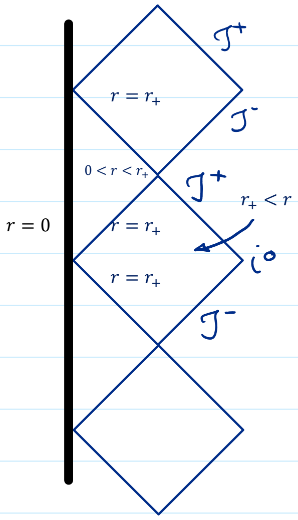
"Strong Cosmic Censorship Conjecture"

There's good evidence that SCC holds, at least when the backreaction of quantum fields is taken into account.

- $M = Q$: extremal RN black hole

$$1 - \frac{2M}{r} + \frac{M^2}{r^2} = \left(1 - \frac{M}{r}\right)^2 \qquad r_+ = r_- = M = Q \quad \text{: double root}$$

$\kappa_+ = 0$: degenerate horizon



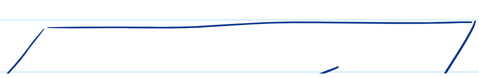
$r = r_+$: event horizon and Cauchy horizon

$r = 0$: timelike singularity

Proper radial distance to $r = M$ is infinite

$$\int_{r>M}^{M} \frac{dr}{1 - \frac{M}{r}} = \int_{r>M}^{M} \frac{r\,dr}{r - M} \quad \text{: diverges at } r = M$$

$$ds^2\big|_t = \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} + r^2 d\Omega \longrightarrow$$



$r \to \infty$: asymptotically flat

Black holes VI: Laws of BHs and BH entropy

jueves, 1 de agosto de 2024 15:47

For RN black holes

$$ds^2 = -V(r)\,dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_2 \qquad V = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

$$A_t = -\frac{Q}{r} + \Phi \qquad \Phi = \frac{Q}{r_+} \qquad \text{so } A_t(r=r_+) = 0$$

$$\mathcal{A}_H = 4\pi r_+^2 \qquad r_\pm = M \pm \sqrt{M^2 - Q^2}$$

$$\kappa = \frac{\sqrt{M^2-Q^2}}{r_+^2}$$

$$\boxed{dM = \frac{1}{8\pi}\kappa\, d\mathcal{A}_H + \Phi\, dQ}$$ 1st law

We may use r_+, r_- as parameters

$$Q^2 = r_+ r_-$$
$$M = \frac{r_+ + r_-}{2} = \frac{r_+ + Q^2/r_+}{2} = \frac{r_- + Q^2/r_-}{2}$$

$$\left(\frac{\partial M}{\partial A_+}\right)_Q = \left(\frac{\partial M}{\partial r_+}\right)_Q \frac{\partial r_+}{\partial A_+} + \left(\frac{\partial M}{\partial r_-}\right)_Q \overbrace{\frac{\partial r_-}{\partial A_+}}^{=0} = \frac{1}{2}\left(1 - \frac{Q^2}{r_+}\right)\frac{1}{2\sqrt{4\pi A_+}}$$

$$= \frac{1}{16\pi}\frac{r_+^2 - Q^2}{r_+^3} = \frac{1}{8\pi}\frac{r_+ - r_-}{2r_+^2} = \frac{\kappa_+}{8\pi}$$

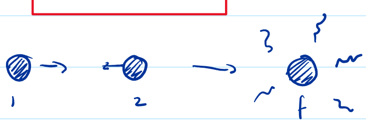
$$\left(\frac{\partial M}{\partial Q}\right)_{A_+} = \left(\frac{\partial M}{\partial r_-}\right)_{A_+}\left(\frac{\partial r_-}{\partial Q}\right)_{A_+} = \frac{1}{2}\,\frac{2Q}{r_+} = \frac{Q}{r_+} = \Phi_+$$

2nd Law (Hawking)

The area of the horizon(s) cannot decrease in time

$$\boxed{\Delta\mathcal{A}_H \geq 0}$$

eg



$$A_1 + A_2 \leq A_f$$

0th Law : $\boxed{\kappa \text{ is uniform over horizon}}$

Striking analogy:

Laws	Thermodynamics	BH mechanics
0^{th}	$T=$ const at equilibrium	$\kappa=$ const over stationary horizon
1^{st}	$dE = TdS - pdV + \Omega dJ + \Phi dQ$	$dM = \frac{\kappa}{8\pi G} dA_H + \Omega dJ + \Phi dQ + \dots$
2^{nd}	$\Delta S \geq 0$	$\Delta A_H \geq 0$

But their derivations couldn't be more different!

Carnot cycles — Null geodesics

Heat reservoirs — Killing vectors, surface integrals

s

Thermodynamics in the presence of black holes looks bizarre.
Could dump entropy into a bh and lose it from sight, entirely and forever!
[Could also extract the energy of a box w/ radiation (Geroch process) and then dump the entropy into the black hole]

Bekenstein: The area of a black hole is a form of entropy

$$S_{BH} = C \frac{A_H}{G\hbar} \sim \frac{A_H}{\ell_{Pl}^2} \qquad \sim 1 \text{ bit per unit Planck area}$$

Generalized 2^{nd} law: $\Delta(S_{matter} + S_{BH}) \geq 0$

There should be a temperature $T_H = \frac{\hbar\kappa}{8\pi C}$ $\qquad \frac{\kappa}{8\pi G} A_H = TS$

Puzzle:

If black holes have entropy, they should have a temperature. If they have a temperature, then they should be able to emit thermal radiation.
But black holes do not emit anything <u>at all</u>. They're at absolute zero temperature! Or are they?

Hawking set out to prove that Bekenstein was wrong: a quantum system in an equilibrium black hole spacetime should be absolutely cold.

[But he actually proved the opposite! When considering a collapsing system, initially a quantum field is parametrically excited by the dynamical geometry. But when it settles down into a stationary equilibrium black hole, the field continues to be excited, and actually there's a continuous geometry. But when it settles down into a stationary equilibrium black hole, the field continues to be excited, and actually there's a continuous outflux of radiation with a thermal spectrum with a temperature

$$T_H = \frac{\hbar \kappa}{2\pi}$$

This determines $C = \frac{1}{4}$ and therefore fixes

$$S_{BH} = \frac{A_H}{4G\hbar} \quad : \text{Bekenstein-Hawking entropy}$$

This formula combines gravitation, quantum theory, and thermodynamics in a deep and mysterious way. It's our first formula for quantum gravity, and our most important clue into a quantum theory of gravity.

Black holes VII: Hawking radiation

jueves, 1 de agosto de 2024 15:48

From 2306.11139 Sec.4

4. The black hole that evaporates

In 1974 Hawking studied quantum fields propagating on a geometry that collapses to form a black hole. He found out that during the process of collapse, quantum radiation is emitted, as expected in a time-dependent situation. But more surprisingly, after the black hole has settled into a stationary configuration and the transient effects have died out, there remains a steady outflow of radiation, which observers at a large distance detect as a black body spectrum[17] with temperature

$$T_H = \frac{\hbar\kappa}{2\pi}, \tag{4.1}$$

4.1 Particle production in an external field

We intend to study the production of particles in a given background field, due to fluctuations of a quantum field. Virtual quantum fluctuations of a field with mass m are described by

$$\langle\phi(x)\phi(0)\rangle \sim e^{-x\frac{mc}{\hbar}}. \tag{4.2}$$

This is the probability amplitude that a pair of field-quanta separated by a distance x would spontaneously form. In vacuum, these fluctuations do not materialize in the production of a real pair since that would violate energy conservation. But the energy required for this materialization can be provided if there is an external field to which the quantum field couples.

Let us denote the field strength (force per unit charge) by F, and the coupling (charge) by λ. Assuming the field is uniform, in order to materialize the pair of quanta, we need the equality

$$\lambda F x = 2mc^2, \tag{4.3}$$

for energy conservation to hold. Then the probability for a pair creation per unit volume and unit time is given by

$$\Gamma \sim |\langle\phi(x)\phi(0)\rangle|^2 \sim e^{-\frac{4m^2c^3}{\hbar\lambda F}}. \tag{4.4}$$

A proper calculation in quantum field theory involves a tunneling process (hence the exponential suppression) which can be evaluated in the WKB approximation, and indeed gives

$$\Gamma \sim A\, e^{-\gamma\frac{\pi m^2 c^3}{\hbar\lambda F}}, \tag{4.5}$$

where A is the quantum one-loop determinant factor, which we will ignore in the following (it is not easy to compute, and yields subdominant corrections), and γ is a numerical factor of order one, which depends on the specific type of particle and its coupling to the field.

This is a process of pair creation by a background field, where the latter is a semi-classical, coherent state involving a large number of quanta in the background. The process is described in terms of a

This is a process of pair creation by a background field, where the latter is a semi-classical, coherent state involving a large number of quanta in the background. The process is described in terms of a non-perturbative instanton bounce. It is different than the perturbative process of pair creation, e.g., e^+e^- creation by photon-photon collision. The non-perturbative e^+e^- production in a background electric field was studied by Schwinger in a classic paper in 1950. In this case $\lambda = e$, $F = E$, and $\gamma = 1$. Schwinger's leading order result yields

$$\Gamma_{e^+e^-} \sim A\, e^{-\frac{\pi m^2 c^3}{\hbar e E}} . \tag{4.6}$$

The energy for the creation of the pair is provided by the background field, which as a result decays gradually.

A black hole creates a strong gravitational field, so we might also expect particle pairs to form near the horizon. In this case the coupling is the particle's mass, while for the force we take the surface gravity,

$$\lambda = m\,, \qquad F = \kappa\,. \tag{4.7}$$

With these choices, we find that

$$\Gamma \sim e^{-\gamma \frac{\pi m c^3}{\hbar \kappa}} . \tag{4.8}$$

Observe that the exponent is proportional to m, i.e., to the energy $E = mc^2$ of the particle. Thus we can write it as

$$\Gamma \sim e^{-E/T_H}\,, \tag{4.9}$$

where

$$T_H = \frac{\hbar \kappa}{\gamma \pi c} . \tag{4.10}$$

This is a *thermal spectrum with temperature* $T_H \propto \kappa$. So, the black hole is expected to radiate like a blackbody. In contrast, the Schwinger production rate (4.6) is not thermal. The reason that it is thermal in the case of a black hole is that gravity couples to the particle's energy. It is very suggestive that the universal character of gravity appears to be related to a universal thermal behavior.

Some caveats about this heuristic argument:

- We have not pinned down the value of γ. For this we need Hawking's proper calculation, which yields $\gamma = 2$.
- The argument was made for massive particles. However, Hawking's result applies as well to massless quanta, with $E = \hbar\omega$.
- The energy required by the creation of the pair is supplied by the black hole. The energy of the black hole decreases, since one of the members of the pair has negative energy (relative to asymptotic observers) and falls inside the black hole. In more detail, if the pair have four-momenta p_1 and p_2, four-momentum conservation requires that

$$p_1 + p_2 = 0\,. \tag{4.11}$$

If ∂_t is the generator of asymptotic time-translations, so that t is conjugate to the energy measured by asymptotic observers, then the energy of a particle with four-momentum p is

If ∂_t is the generator of asymptotic time-translations, so that t is conjugate to the energy measured by asymptotic observers, then the energy of a particle with four-momentum p is

$$E = -p \cdot \partial_t \, . \tag{4.12}$$

Thus for the particle pair we must have $E_1 + E_2 = 0$. Now, if particle 1 is to escape to infinity, it must have $E_1 > 0$. If particle 2 goes inside the black hole, then in that region ∂_t is spacelike, so E_2 is actually not an energy but a component of momentum, which can be negative. Thus it is consistent to create the pair if one of the particles falls inside the black hole. In this case, the total mass of the black hole will decrease by an amount

$$\delta M = E_2 = -E_1 \, , \tag{4.13}$$

consistent with conservation of the energy as measured by outside observers.

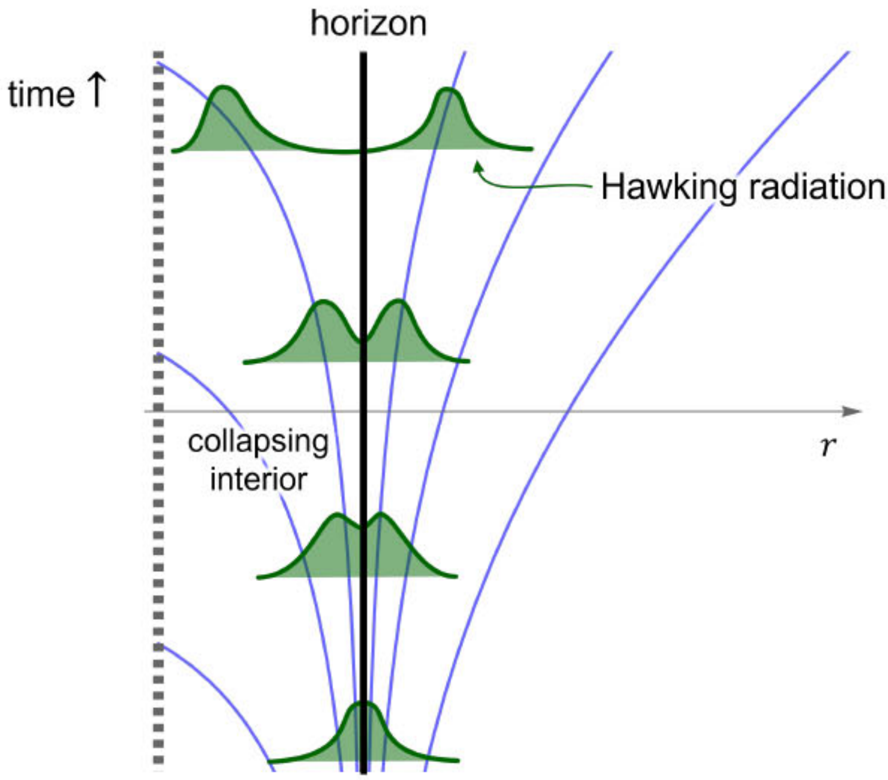


Figure 17: Hawking emission as quantum pair production, in an Eddington-Finkelstein diagram (cf. fig. 2). A quantum wavefunction initially straddling the horizon is stretched by the collapse of the black hole interior, until one of its components escapes away as a Hawking quantum and the other falls to the singularity. The pair is created in an entangled Bell-like state.

Let us provide yet another viewpoint on why quantum physics in the presence of a black hole horizon gives rise to what may seem an unexpected consequence. A classical particle outside a stationary black hole perceives that nothing changes. A quantum field can be excited when the geometry that it lives in is time-dependent, a phenomenon of parametric excitation well-known in cosmology. The exterior of the black hole is static, and so naively one might conclude that no field excitation will occur. However, a quantum wave function can have support simultaneously in the exterior and the interior of the black hole. This is a crucial property, since the interior geometry of a black hole is dynamically collapsing (a Little Big Crunch) in a time scale $\sim \kappa^{-1}$ (in units $c = 1$). A quantum field will be sensitive to the time dependence of the collapsing interior and therefore will be excited. The characteristic time of this collapse implies that the field excitations will have typical frequency $\sim \kappa$, and the energy of the produced quanta will be $\sim \hbar\kappa$, which agrees with (4.10).

The part of the wave function that is in the interior will be dragged in by the collapse, and stretched until it becomes a wave packet separated from its exterior partner (see figure 17). This produces a pair of quanta, with positive and negative energies relative to asymptotic time. The quantum that is radiated away will be entangled with the quantum in the interior, since they are part of the same wave function, so they will be in a maximally entangled Bell-like state.

Observe also that, according to this interpretation, a black hole with a time-independent interior (static or stationary) must not give rise to Hawking radiation. This is indeed what happens in extremal black holes, which have $\kappa = 0$ and therefore $T_H = 0$ too. Nevertheless, extremal black holes can decay through non-thermal spontaneous emission of superradiant modes (3.18).

4.2 Further aspects of Hawking radiation and black hole evaporation

We can now draw two major consequences: black holes must have entropy, and they will evaporate.

Back to the 19th-century, with a black hole. Rudolf Clausius explains to us that black holes must be assigned an entropy on very general phenomenological grounds: an object with energy E that radiates at a temperature T has an entropy S given by

$$S = \int \frac{dE}{T} . \tag{4.14}$$

Let us see how this works. As the seasoned experimentalists that we are, we can measure the energy (mass) of the black hole and its temperature from far away without even knowing that the radiating object is a black hole. We proceed to collect data using a dynamometer and a bolometer and thus we obtain $T(M)$. Our measurements yield

$$T = \frac{\hbar}{8\pi GM} , \tag{4.15}$$

which, using that $E = M$, we plug into (4.14) to find

$$dS = \frac{8\pi G}{\hbar} M \, dM , \tag{4.16}$$

$$\Rightarrow \quad S = \frac{4\pi G}{\hbar} M^2 , \tag{4.17}$$

When we work out the numbers we are at first surprised to find that this entropy is (as we will see in a moment) enormous. But we attribute it to having a peculiar object that has a very large energy but is radiating at an extremely low temperature.

By now we have also figured out, e.g., by scattering particles or waves, that the radiating object is a black hole whose radius we have measured to be $r_H = 2GM$. Then we realize that we can write

$$S = \frac{4\pi(2GM)^2}{4G\hbar} = \frac{\mathcal{A}_H}{4G\hbar} \tag{4.18}$$

$$= \frac{\text{Horizon area}}{\text{Planck area}} . \tag{4.19}$$

Therefore, the entropy of the black hole is nothing but its area measured in Planck units—truly, an enormous number for any macroscopic area! But then Clausius looks puzzled at us: why should entropy—a quantity that he introduced for understanding the efficiency of heat exchanges—bear any relation with that most basic entity, the geometry of space and time?

The identification of the black hole area with an entropy was first proposed—very boldly, and not without controversy—in 1973 by Bekenstein. It was then put on a firm footing by Hawking's discovery that black holes emit thermal radiation with a precise temperature. For this reason,

$$\boxed{S_{BH} = \frac{\mathcal{A}_H}{4G\hbar}} \tag{4.20}$$

is called the Bekenstein-Hawking Black Hole entropy formula. It contains G and $\hbar$, and over the last half-century it has provided the deepest and most fruitful guidance towards a quantum theory of gravity.

Now we go to interrogate other 19th-century physicists about the implications of this result. Ludwig Boltzmann informs us that, ultimately, it implies the existence of many microscopic states corresponding to the macroscopic system that we characterize by this mass and temperature i.e., the black hole. Indeed, he continues, there must be as many as $e^{S_{BH}}$ states! In other words, a black hole—which is nothing but strongly warped space and time—must somehow be made of a humongous number of microscopic degrees of freedom, even if we do not see them at all.

Boltzmann raises an eyebrow and asks: if we are to think of these degrees of freedom as somehow related to 'atoms of spacetime' (an idea he seems to relish), shouldn't their number scale like the spatial volume, instead of the area? He begins to warn us about the tribulations of applying statistical reasoning to entities that have long been regarded as paradigms of determinism, but it is time that we take leave of him (some of his concerns will reappear later) and continue with other consequences of Hawking's discovery.

Evaporation rate and black hole lifetime. The black hole will evaporate by emitting radiation like a blackbody. The radiating power of a blackbody of area A and temperature T is

$$\frac{dE}{dt} = \sigma A T^4 \,, \tag{4.21}$$

where σ is the Stefan-Boltzmann factor, which depends on the specific (effectively massless) fields that are being radiated. For a real scalar field, $\sigma = \pi^2/60$. As a first approximation we can take $A \simeq \mathcal{A}_H$. For a Schwarzschild black hole,

$$\mathcal{A}_H = 16\pi M^2 \,, \qquad T = \frac{1}{8\pi M} \,, \tag{4.22}$$

and since $dE = -dM$ we can obtain the evaporation rate of the black hole as

$$\frac{dM}{dt} \simeq -\frac{\sigma}{256\pi^3}\frac{1}{M^2} \propto M^{-2} \,, \tag{4.23}$$

so that the total evaporation time derived from

$$\int_0^{t_{\text{evap}}} dt \propto -\int_M^0 dM'\, M'^2 \,, \tag{4.24}$$

is simply

$$t_{\text{evap}} \propto M^3 \,. \tag{4.25}$$

It is then obvious that the black hole evaporates in a finite time. This calculation is of course very rough since it neglects the back-reaction effect that the emission of radiation and loss of mass have on the black hole geometry and on the radiation process itself. These effects should be small when the energy of emitted quanta is much smaller than the mass of the black hole,

$$T_H \ll M \,, \tag{4.26}$$

i.e., as long as $M \gg M_{\text{Planck}}$. Therefore, for most of the black hole lifetime the approximation is good, and its mass will reach Planck size in a time $\propto M^3$.

Generalized Second Law (Bekenstein 1972). The second law of thermodynamics is extended to include the contributions from black hole entropy so that

$$\Delta S_{\text{total}} = \Delta S_{BH} + \Delta S_{\text{matter}} \geq 0\,. \tag{4.31}$$

Each of the two contributions can separately be negative, but only as long as their total sum is positive.

There is a 2nd law of black holes, proven by Hawking, that applies in the classical theory:

$$\Delta A_H \geq 0.$$

"BH area Theorem": The total area of the event horizon cannot decrease in time.

The GSL reduces to this in the limit $\hbar \to 0$

since $S_{BH} = \frac{A_H}{4G\hbar} \propto \frac{1}{\hbar}$ while $S_{\text{matter}} \propto \hbar^0$

so

$$S_{TOT} = \frac{A_H}{4G\hbar} + S_{\text{matter}} \xrightarrow{\hbar \to 0} \frac{A_H}{4G\hbar}$$

In the process of collapse to a black hole, $\Delta S_{\rm matter} < 0$ but the entropy S_{BH} that is produced overwhelms this decrease by far. A simple estimate for the collapse of a weakly-gravitating ball of radiation gives

$$\frac{\Delta S_{BH}}{|\Delta S_{\rm matter}|} \sim \left(\frac{M}{M_{\rm Planck}}\right)^{1/2} > 1\,, \tag{4.32}$$

which is in fact $\gg 1$ for any astrophysical black hole. In order to understand the huge increase in the entropy, observe that the ordinary matter before the collapse naturally probes only a small, low-energy subset of all the degrees of freedom of the fundamental theory (e.g., string theory), so it has a low entropy. The black hole, instead, samples all the stringy (fundamental) states available.[20]

On the other hand, in the process of black hole evaporation we have $\Delta S_{BH} < 0$, but the radiation carries sufficient entropy to compensate for it. Again a simple calculation for the entropy production when massless radiation is emitted can be seen to yield

$$\frac{\Delta S_{\rm rad}}{|\Delta S_{BH}|} \simeq \frac{4}{3} > 1\,. \tag{4.33}$$

Observe that in this case the entropy produced is of the same order as the entropy of the black hole. The reason is that the evaporation produces a number of quanta $N \sim S_{BH}$. The factor $4/3$ comes from the relation $s_{\rm rad} = \frac{4}{3}\epsilon_{\rm rad}T$ between the entropy and energy density of massless radiation at temperature T (in D spacetime dimensions, $s_{\rm rad} = \frac{D}{D-1}\epsilon_{\rm rad}T$). The number is guaranteed to be > 1 for any form of radiation or matter that is thermodynamically stable.

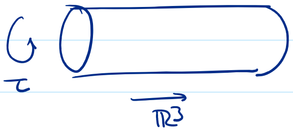
Black holes VIII: BHs in Euclidean Quantum Gravity

jueves, 1 de agosto de 2024 15:48

Black holes in imaginary Time

For studying QFT at finite Temperature we consider The QFT in

$$ds^2 = \underbrace{d\tau^2}_{S^1} + \underbrace{dx^2 + dy^2 + dz^2}_{\mathbb{R}^3} \qquad \tau \sim \tau + \beta \qquad S^1 \times \mathbb{R}^3$$



What does This have To do with bhs?

$$t = -i\tau$$

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega$$

$\tau \sim \tau + \beta$: we're going To see That not all β are allowed

Go near The "Euclidean horizon" at $r = 2M$

$$r = 2M + \xi^2 \qquad \xi^2 << 2M$$

$$ds^2 \simeq \frac{\xi^2}{2M} d\tau^2 + 8M\, d\xi^2 + 4M^2 d\Omega$$

$$8M\,\xi^2 = x^2$$

$$ds^2 \simeq +\kappa^2 x^2 d\tau^2 + dx^2 + 4M^2 d\Omega \qquad \kappa = \frac{1}{4M} \text{ : surface gravity}$$

$$\rho^2 d\phi^2 + d\rho^2 \qquad x = \rho$$

$$\kappa\tau = \phi \qquad \tau = \frac{1}{\kappa}\phi$$

$$\tau \sim \tau + \beta \qquad \phi \sim \phi + \kappa\beta$$

Normally $\phi \sim \phi + 2\pi$. If instead $\kappa\beta < 2\pi$: conical singularity

$\kappa\beta = 2\pi$ for absence of conical singularities in The Euclidean geometry

$$\Rightarrow \quad \beta = \frac{2\pi}{\kappa} \qquad T = \frac{\kappa}{2\pi} \qquad T = \frac{\hbar\kappa}{2\pi} \text{ : Hawking Temperature}$$

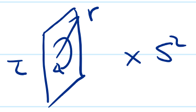
Since The geometry is periodic $\tau \sim \tau + \beta$, all correlation functions of

Since the geometry is periodic $\tau \sim \tau + \beta$, all correlation functions of a QFT in this geometry will also exhibit this same periodicity

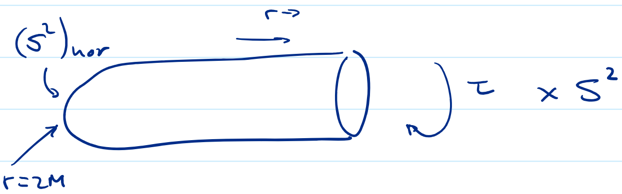
$\Rightarrow$ QFT in a bh background is finite-temperature field theory at $T = T_H = \frac{\kappa}{2\pi}$

The geometry of Euclidean Schwarzschild:

near $r = 2M$



large r $\quad ds^2 \to d\tau^2 + dr^2 + r^2 d\Omega$

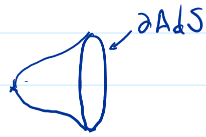


There's nothing behind $r = 2M$

Geometry is completely non-singular and geodesically complete

Topology is $\mathbb{R}^2 \times S^2 \neq S^1 \times \mathbb{R}^3$

This is also valid for black holes in AdS



BH entropy from Euclidean Quantum Gravity (Gibbons + Hawking)

$$Z[\beta] = \int \mathcal{D}g \, e^{-I_E[g]}$$

$I_E[g]$: classical Euclidean action

sum over geometries that are periodic in imaginary time $\Delta\tau = \beta$

The Euclidean gravitational action is

$$I_E = -\frac{1}{16\pi G}\int_M d^4x \sqrt{g}\, R - \frac{1}{8\pi G}\int_{\partial M} d^3x \sqrt{h}\, K$$

and it is crucial to include the GHY boundary term.

GPI is ill-defined: - UV problems: non-renormalizability
- IR problems: unboundedness under conformal transformations (there are ways around this)

- what geometries / Topologies / signatures... are allowed?

Evaluate it in the <u>saddle-point approximation</u>:

$$Z[\beta] \approx e^{-I_E[g_{cl}]}$$

g_{cl} : classical solution to the field equations of $I_E[g]$ $\quad \frac{\delta I_E}{\delta g}[g_{cl}]=0$

with g_{cl} periodic in $\tau \sim \tau + \beta$

This is like a WKB approximation to quantum mechanics, in the semiclassical limit.

There may be more than one saddle contributing. The dominant one is the one with smallest $I_E[g_{cl}]$.

Once we have computed the partition function w/ canonical boundary conditions (fixed Temperature), we can apply conventional Thermodynamics to it:

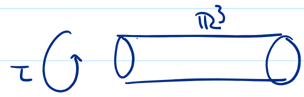
$$F = -\frac{1}{\beta}\log Z \approx \frac{1}{\beta} I_E \quad \text{(using the saddle-point result)}$$

$$E = -\frac{\partial \log Z}{\partial \beta} \approx \partial_\beta I_E$$

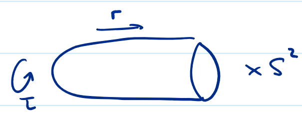
$$S = -(\beta\partial_\beta - 1)\log Z \approx (\beta\partial_\beta - 1) I_E$$

For AF spaces w/ $\tau \sim \tau + \beta$, we have at least two different saddle point geometries:

Euclidean flat space $S^1 \times \mathbb{R}^3$



Euclidean Schw black hole $S^2 \times \mathbb{R}^2$



↳ horizon is a fixed point of Euclidean Time evolution

When computing $I_E[g_{cl}]$, the GHY term is crucial. E.g. for a vacuum solution only the boundary term contributes.

The action I_E is infinite, since space is infinite. To deal with this:

- First, regularize w/ cutoff at $r = R$ (large radius)
- Subtract $I_E - I_E^0$ I_E^0 action of a reference background state (~vacuum)

  Ensure that boundary conditions are the same for $g_{\mu\nu}$ and $g^0_{\mu\nu}$.

$$g_{\mu\nu}: \text{ Euc Schw with } \beta = 8\pi M$$
$$g^0_{\mu\nu}: \text{ Eucl Mink}$$

$\Rightarrow$ finite $I_E - I_E^0 \approx \beta F(\beta)$ for Schw bh

This can indeed be carried out and one finds $F = \frac{\beta}{16\pi}$

so, using the expressions above, $E = \frac{\beta}{8\pi} = M$ which are correct.

$$S = 4\pi M^2 = \frac{A_H}{4}$$

In AdS, it is possible to perform the subtraction without reference to a background. One introduces "boundary counterterms", which are of the form $\int_{\partial M} \sqrt{h}\, F(g, \nabla g \ldots)$

$\hookrightarrow$ local intrinsic invariants of the bdy geometry
eg $\sqrt{h}$, $\sqrt{h}\, R(h)$, ...

(In AF, counterterms are <u>not local</u> invariants).

There is a general argument to show that in any Einstein-Hilbert theory of gravity (w/ any matter), the GPI yields an entropy

$$S = \frac{A_H}{4G\hbar}$$

In this sense, there's a <u>very direct relation between Einstein gravity</u> (ie the EHGHY action) <u>and $S \propto A_H$</u>.   "Matter = Geometry"

There's a general formula for the entropy of higher-curvature gravities due to Wald, which generalizes Bekenstein-Hawking.

This entropy is due to the non-trivial topology of the Euclidean black hole solution, specifically to the fixed point set of the isometry $\partial/\partial\tau$.

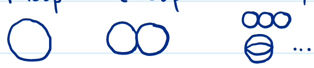
- This is a <u>classical contribution to entropy</u>

In QFT $\quad \sum_i \int d^4p$: sum over states running in the circle

The action receives contributions in a loop expansion

$$I = \hbar + \hbar^2 + \hbar^3 + \dots$$

$\uparrow$ 1-loop $\quad \uparrow$ 2-loop $\quad \uparrow$ 3-loop



In QFT at finite β, we have $F, S \sim \frac{I}{\hbar}$

so $\quad S \propto \hbar^0 + \hbar + \hbar^2 + \dots$

But here we have found

$S_{BH} = \frac{A_H}{4G\hbar}$: 0-loop calculation of entropy?!

what microscopic states are being counted?

How does the gravitational path integral give us the correct result for the entropy, while not performing any sum over microscopic states?

<u>In string theory</u>, it has been possible to perform countings over microstates of a fundamental theory that correctly reproduce the BH entropy:

$S_{micro} = S_{BH} \qquad S_{micro}$: stat-mech counting of entropy

Build a susy bh w/ certain charges, mass, and area $S_{BH} = \frac{A_H}{4G}$

It's very similar to the extremal Reissner-Nordstrom solution,

for which $\quad S_{BH} = \pi Q^2$

Now consider the gravitational coupling G to be variable.

$G \to 0$: The size of horizon decreases

until its small enough that it doesn't look like

a black hole, but a configuration of strings (on D-branes)

As we change g : mass changes (renormalization of mass)

Keep charges fixed

susy $\Rightarrow$ entropy as a function of charges doesn't change

In the stringy side we can do a stat-mech calculation of S_{micro}.

Strominger + Vafa (1996) found $S_{micro} = S_{BH}$

The result has been generalized and extended for large classes of black holes, in different dimensions, with different kinds of charges, different horizon topologies, and sometimes with higher-curvature corrections.

4.3 The black hole information loss problem

Shortly after his discovery that black holes emit quantum thermal radiation, Hawking realized a surprising consequence: black hole evaporation implies that classical general relativity and quantum mechanics are incompatible.

It was already known that the merger of the two theories was marred by the non-renormalizability of the quantum theory of the massless spin-two field (the graviton). But the latter is a problem of the ultraviolet, high-energy structure of the theory and becomes manifest only when trying to probe the nature of spacetime close to the Planck scale, where the curvatures are so large that the very notion of spacetime geometry becomes dubious. It is therefore a situation where we naturally expect that general relativity, regarded as an effective theory, breaks down.

The problem that Hawking identified is much more surprising, since it appears in situations where the geometry of spacetime is expected to be smooth. The curvature near the horizon of a black hole scales inversely to its size, so for a very large black hole the region around the horizon can be as weakly curved as we wish. Hawking's problem arises for semiclassical black holes with sizes arbitrarily larger than the Planck length. Since there is no diff-invariant quantity that diverges in these regions, why should the effective theory break down?

This has turned out to be a surprisingly deep and subtle question—and a very confusing one too. We will not be able to do proper justice to all its intricacy, but we will attempt to make the reader aware of its gist, without going into the resolutions that have been proposed.

4.3.1 Classical prelude: Black holes have no (or too little) hair

A main property of black holes, mentioned in the previous sections, is that they are characterized by very few parameters. The black hole phase space is therefore of low dimensionality.

We referred to this by saying that 'black holes have no hair': the space of black hole solutions is completely characterized by the conserved charges of the black hole, namely its mass, angular momentum, and charges associated to local conservation laws such as electric charges. Nowadays we know of the existence of many kinds of black hole hair. Nevertheless, the phase space of black hole solutions is still characterized by a number of parameters that, for all we know, is much smaller than the huge number $\mathcal{A}_H/\ell^2_{\text{Planck}}$ that the Bekenstein-Hawking formula suggests.

Therefore, when a system collapses to form a black hole, it appears that almost all of the initial information of the configuration of collapsing matter is lost in the process. The final state only knows about a few aggregate, macroscopic quantities. Classically, there is no way that this information can be retrieved. Therefore the black hole appears to be a sink where phase-space volume is destroyed.

4.3.2 Basic formulation of the information loss problem

One may still regard this as not truly problematic, because the information about the state of the matter that formed the black hole may somehow be preserved inside the black hole. However, if the black hole evaporates and disappears then the problem comes back[21]. We will formulate the puzzle in different manners, each at a different level of sophistication and detail. The first formulation contains the core of the problem. The next one makes it more precise. Further formulations are possible that highlight other subtleties of the problem, but we will not discuss them here.

Fundamental irreversible evolution. The original configuration of matter may have been arbitrarily complicated, needing a huge number of parameters to describe it. All but a handful of these data are lost to the outside observers when the black hole forms. The black hole evaporates by emitting Hawking radiation, but this radiation, having a Boltzmann spectrum, only carries information about the total mass of the black hole, and about the conserved charges conjugate to 'chemical potentials' such as angular velocities, electric potentials etc. Thus, when the black hole finally disappears, the initial information is lost.

[21] There is a version of the information loss problem that does not involve the collapse and evaporation of a black hole, but instead deals with the very-long-time behavior of fluctuations in the system of a black hole in equilibrium with its radiation. Significant progress has recently been made in this direction, but we will not discuss it in these notes.

This is a problem of fundamental irreversibility at a deeper level than statistical thermodynamic irreversibility: from the final radiation state we cannot reconstruct, *even in principle*, the initial matter state.

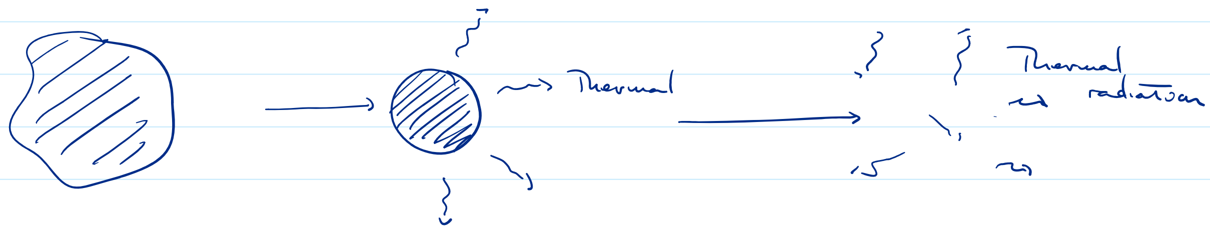
Non-unitary quantum evolution. In the previous paragraph we used 'information' in a rather loose sense. Let us now make it more precise by phrasing the previous problem in quantum mechanical parlance. States in quantum mechanics can be generically described using density matrices ρ. For pure states, which correspond to vectors $|\Psi\rangle$ (properly, rays) in a Hilbert space, the matrix $\rho_{\text{pure}} = |\Psi\rangle\langle\Psi|$ has von Neumann entropy $-\text{Tr}\rho_{\text{pure}} \log \rho_{\text{pure}} = 0$. Mixed states are characterized by having instead $-\text{Tr}\rho_{\text{mixed}} \log \rho_{\text{mixed}} > 0$. Thermal ensembles $\rho_{\text{th}} = Z^{-1} \sum_i e^{-E_i/T} |i\rangle\langle i|$ are of this kind and in fact maximize the entropy. Unitary evolution in quantum mechanics takes a pure state into another pure state

$$|\Psi\rangle \to U(t)|\Psi\rangle\,, \qquad \rho_{\text{pure}} \to \rho_{\text{pure}}\,, \tag{4.34}$$

($U^{-1} = U^\dagger$) and therefore no (fine-grained) entropy can be generated. This is the quantum-mechanical counterpart of Liouville's theorem that Hamiltonian evolution conserves phase space volume, i.e., there are no sinks in phase space. Unitary evolution is also the quantum-mechanical statement that the initial state can be reconstructed out of the final state. So unitarity can be regarded as implementing the notion of fundamental reversibility of quantum-mechanical evolution.

We may take the initial state of the matter that will collapse into a black hole to be a pure state. During most of the evaporation the radiation is thermal to a very good approximation. When the evaporation is completed, only this radiation remains. Thus the evolution of the system violates unitarity, which is one of the basic assumptions of quantum mechanics. The 'black hole information problem' is now stated as the 'black hole unitarity problem'.

Imagine that we manage to collect all the radiation and all the final products of the evaporation of the black hole, isolating them so well as to prevent any losses of coherence to external systems. We pass these black hole remains through a quantum super-computer (still under construction) which attempts to reconstruct the initial state, that is, it tries to find a unitary transformation that, acting on the final state, would yield back the initial one. Hawking affirmed that the quantum computer will fail.



$|\Psi\rangle$

$\rho_{\text{pure}} = |\Psi\rangle\langle\Psi|$ $\qquad$ ρ_{thermal} : mixed state

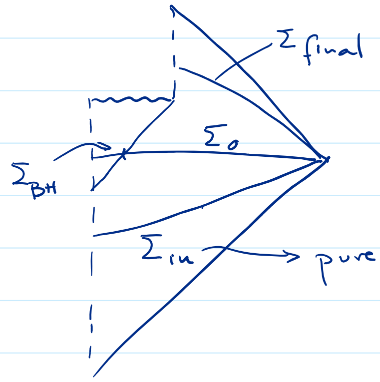
$S = 0$ $\qquad$ pure $\longrightarrow$ mixed $\qquad$ $S = -\text{Tr}\,\rho \log \rho > 0$

But in QM: $\rho(t) = U(t)\,\rho(0)\,U^\dagger(t)$

pure $\to$ pure

Hawking (1976): evolution in quantum gravity is not unitary

"Nice slice" argument:



State in Σ_0 is entangled with the state in Σ_{bh}

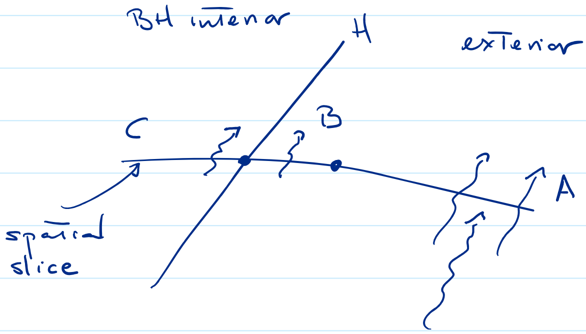
$\downarrow$

Mixed state in Σ_0 is purified w/ system in Σ_{bh}

State in Σ_{final} is determined only by the state in Σ_0

$\Rightarrow$ State in Σ_{final} is mixed

Entanglement structure and strong subadditivity in black hole evaporation



A : first half of the early radiation

B : later Hawking modes near the horizon

C : inside partner modes near horizon

$R = ABC$ is pure

A, B, C are disjoint subsystems

Now we make some assumptions about this system:

(1) Quantum Theory applies to the entire system
("The rules of quantum field theory in curved spacetime apply")

Then the following inequality must hold (SSA):

Strong subadditivity (weak monotonicity)

$$S(AB) + S(BC) \geq S(A) + S(C) \quad (1)$$

The idea is that a given qubit in B can be entangled with A, reducing $S(AB)$, or entangled with C, reducing $S(BC)$, but not both.
This is sometimes called "monogamy of entanglement".

There are equivalent formulations of it,

$$S(AB) + S(BC) \geq S(B) + S(ABC) \quad *$$

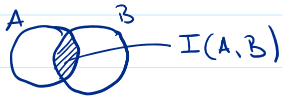
or $\quad S(B|A) + S(B|C) \geq 0$

(where $S(B|A) = S(AB) - S(A)$)

$*$: purify w/ D $\quad S(ABCD) = 0$
$S(ABC) = S(D)$
$S(AB) = S(CD)$
Then $S(CD) + S(BC) \geq S(B) + S(D)$
Now rename $C \to B$, $D \to A$, $B \to C$

It can also be expressed in terms of the mutual information between two systems, defined as

$$I(A,B) = S(A) + S(B) - S(AB)$$



It quantifies how much you can know about B by knowing only A, and viceversa.

Then: subadditivity is $I(A,B) \geq 0$ (This is conventional subadditivity of thermodynamic entropy)
strong " is $I(A,BC) \geq I(A,B)$: knowing both B and C will tell you about A at least as much as knowing only B
equiv $I(C,AB) \geq I(C,B)$
(This may seem obvious for classical information, but it is also true for quantum information)

(2) Unitary evaporation requires that

$S(AB) < S(A)$ (2) : $\exists$ entanglement between B and A, so that information is collected in the radiation in the asymptotic region

Unitarity: The radiation that is collected at infinity contains enough information to reconstruct the initial state

(3) Smoothness of the horizon requires that C purifies B:

$$S(BC) < S(C) \quad (3)$$

This is because Hawking radiation originates in Bell-like pairs which are entangled w/ each other, one of them in B, the other in C.
This is the structure of the state of a quantum field that is regular across a horizon. We saw it for a Rindler horizon: The Minkowski vacuum (regular across the horizon) is fully entangled when the system is divided in two by the horizon. And this structure is locally valid for all smooth horizons. If we break the entanglement between the two sides of the horizon, the state of the quantum field will not be

all smooth horizons. If we break the entanglement between the two sides of the horizon, the state of the quantum field will not be regular: the stress tensor will diverge at the horizon, and this won't be smooth anymore.

The paradox is that (1), (2), (3) are incompatible.

Firewall proposal: Keep SSA + Unitarity. Then the horizon can't be smooth

Information loss: Keep SSA + smooth horizon

No-cloning Theorem:

$U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle$: cloning

$U(|\psi\rangle + |\phi\rangle)|0\rangle = (|\psi\rangle + |\phi\rangle)(|\psi\rangle + |\phi\rangle) = |\psi\rangle|\psi\rangle + |\phi\rangle|\phi\rangle + |\phi\rangle|\psi\rangle + |\psi\rangle|\phi\rangle$

$U|\psi\rangle|0\rangle + U|\phi\rangle|0\rangle = |\psi\rangle|\psi\rangle + |\phi\rangle|\phi\rangle \quad \neq$

With an ancilla A:

Cloning: $U|\psi 0 A\rangle = |\psi\psi A'\rangle$: impossible if U is linear

(reverse: no-deleting Thm)

$U(|\psi 0 A\rangle + |\phi 0 A\rangle) = |\psi\psi A'\rangle + |\phi\phi A''\rangle$ (1)

$U(|\psi\rangle + |\phi\rangle)|0A\rangle = (|\psi\rangle + |\phi\rangle)(|\psi\rangle + |\phi\rangle)|A'''\rangle$

$= |\psi\psi A'''\rangle + |\phi\phi A'''\rangle$

$+ |\psi\phi A'''\rangle + |\phi\psi A'''\rangle$ (2)

(1) $\neq$ (2)

Black holes X: Information paradox (II)

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Page curve, black hole unitarity, entanglement islands

Corrections due to backreaction. A better-looking alternative is to imagine that black hole evaporation is not essentially different from the process of burning a chunk of coal, or, conceptually equivalently, gradually annihilating a lump of matter by throwing antimatter at it. We know that this process is governed by quantum mechanics and thus the evolution is unitary. Although in practice it is very difficult to reconstruct the initial state from the final radiation, we know that the information must be there in the form of extremely subtle correlations among radiation quanta. Since we know that Hawking's calculation must at any stage receive small corrections due to the backreaction effects of the radiation on the black hole, it could be that these corrections serve the purpose of encoding the information on the outgoing quanta, in the form of small deviations from the exactly thermal spectrum.

Appealing though this may seem, it can be argued that small deviations in the Hawking emission of each quantum cannot add up to the required total information. In the case of the annihilation of a lump of matter by throwing antimatter, or any initially pure system emitting radiation in a manner governed by a Hamiltonian, the entanglement between the emitted radiation and the burning system initially grows, peaking at the moment when half the total energy has been radiated. From that point on, the entanglement decreases, meaning that the radiation carries more and more information of the total state, and finally the entanglement disappears: all the information is stored in the radiation (although in a highly scrambled manner).

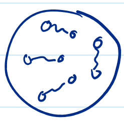
This is not what Hawking emission appears to be doing. The evaporation of a black hole proceeds by creating pairs of quanta. Each member of the pair is entangled with the other: they can be regarded as an EPR pair. One of the two quanta falls into the black hole; the other escapes to infinity. Thus, the entanglement between the interior and the exterior of the black hole grows monotonically along the evaporation. If backreaction results in a small modification in the individual emission process, the entanglement will still keep growing, only now at a slightly lower rate. So it does not seem possible that backreaction, which must be small for large black holes, can effect the reversal in the growth of the entanglement at mid-evaporation that is required to maintain unitarity of the entire process.

Page cuv and black hole unitarity

Consider system made of N Bell pairs

A B

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

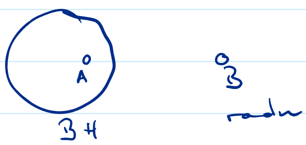


$$\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi| = \frac{1}{2}\left(|0\rangle_A\langle 0| + |1\rangle_A\langle 1|\right)$$

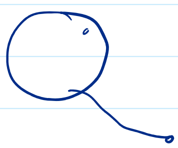
$$p_0 = 1/2 \qquad p_1 = 1/2$$

$$S_A = -\sum_i p_i \log p_i = \log 2$$

It is "evorating" by emitting members of Bell pairs

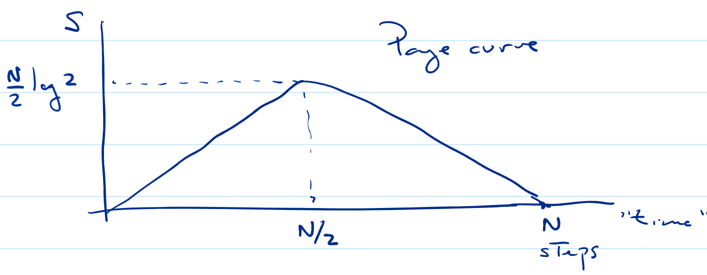


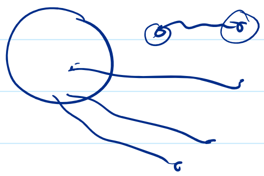
$\Delta S = \log 2$



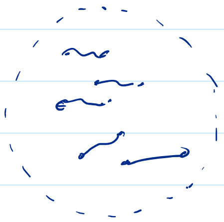
$S = \frac{N}{2}\log 2$

$\Delta S = -\log 2$





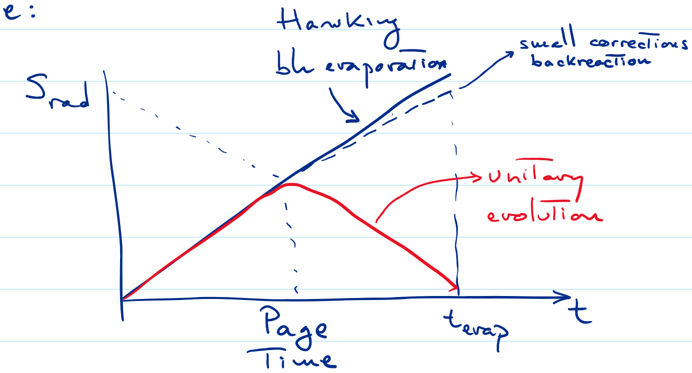
$\Delta S = -\log 2$



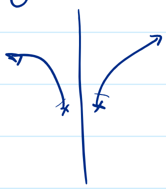
$S = 0$

In an evaporating black hole:

Instead of Bell pairs, we have thermally entangled states, but they're analogous.



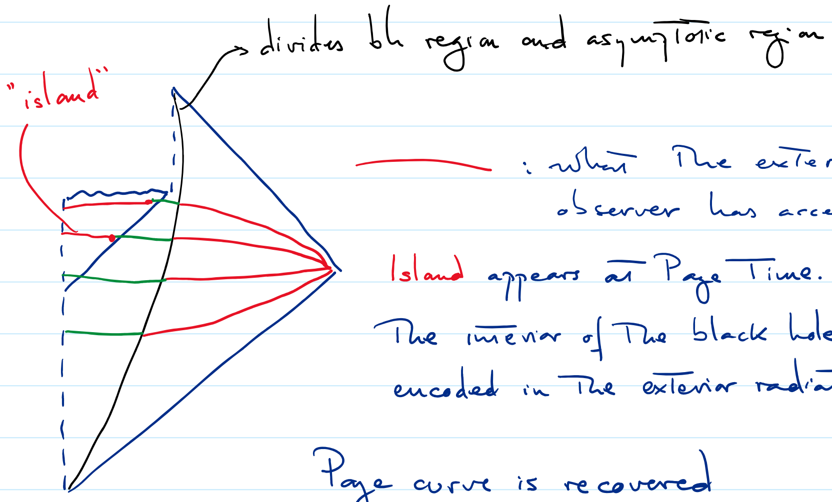
Hawking evaporation:



$\Delta S = \log 2$

After the Page Time, this picture can't hold anymore. The created quantum must be entangled w/ early radiation.

Entanglement islands

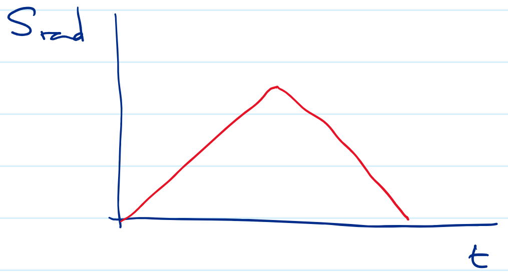


—— : what the exterior observer has access to.

Island appears at Page time.

The interior of the black hole is encoded in the exterior radiation

Page curve is recovered



This has been computed using the euclidean gravitational path integral

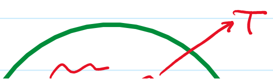
Page curve for eternal black holes

Take a black hole with temperature T, and at $t=0$ put it in contact with a "radiation bath" at the same temperature

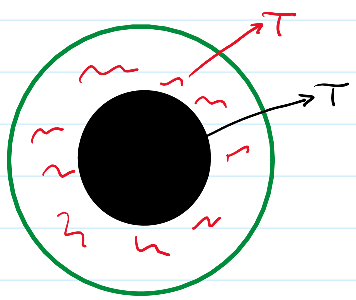
Initially the bath is in a pure state that is very approximately thermal.

Say they're initially separated by a wall that is then removed.

(The entire system is in a box, or in AdS, in such a way that stable thermal equilibrium can be achieved).



The bh and the bath exchange radiation

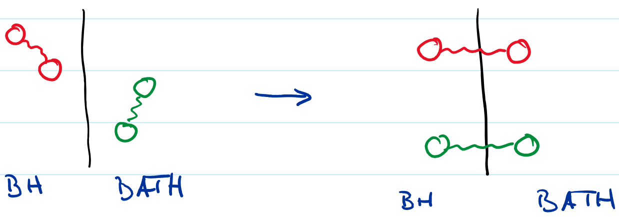


The bh and the bath exchange radiation in detailed equilibrium, in such a way that the bh mass does not change:

radiation flux in = radiation flux out.

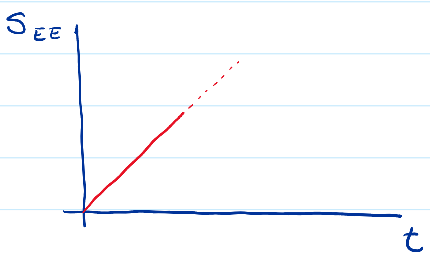
The geometry doesn't change (good simplification!) but the system is evolving in a subtle way: the two subsystems, bh and bath, are becoming increasingly entangled.

The quanta that the bh emits are entangled with interior partners, and the quanta of the bath that go into the bh are also entangled with the outer bath radiation



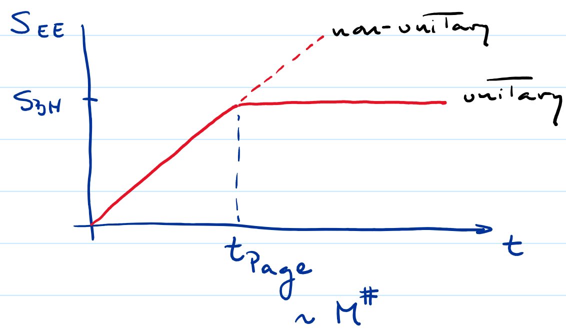
$$\Delta S = 2 \ln 2$$

$$\Delta E = 0$$



entanglement between bath and bh grows linearly in time

If Hawking's picture of evaporation remains correct, then S_{EE} keeps growing w/out bound. But if S_{BH} measures "the number of Bell pairs that the bh consists of", then S_{EE} should saturate when $S_{EE} = S_{BH}$: external radn saturate when $S_{EE} = S_{BH}$: external radn can only be entangled with a number $e^{S_{BH}}$ of internal bh states:



A Page curve of This Type has been recovered by analyzing The problem with The gravitational path integral, incorporating spacetime wormholes. The number of independent internal states of The black hole, constructed and computed w/ The GPI w/ wormholes, is not ∞ but $e^{S_{BH}}$.

This doesn't require string Theory, or a UV-complete quantum Theory of gravity. The GPI does it in a very universal way!

There are other aspects of The BHIP for a black hole in Thermal equilibrium That appear at much later Times, $t \sim e^{S}$

Late-Time correlations and information loss

Study how Two-point correlation functions behave in a system initially in Thermal equilibrium:

$$\langle O(t)\, O(0)\rangle_\beta = \mathrm{Tr}\left(e^{-\beta H}\, O(t)\, O(0)\right)$$

perturb at $t=0$, measure again at t

$$= \sum e^{-\beta E_n} \, |\langle m| O |n\rangle|^2 \, e^{-it(E_m - E_n)}$$

$\langle O(t)\, O(0)\rangle_\beta = \mathrm{Tr}(e^{\cdot}\; O(t)\, O(0))$ measure again at t

$$= \sum_{n,m} e^{-\beta E_n} \left|\langle n|O|m\rangle\right|^2 e^{-it(E_m-E_n)}$$

Sum over discrete spectrum of energies E_n.

Initially, correlations decay $\sim e^{-t/\tau}$

$\tau \propto \beta$: Thermalization Time

Later

$e^{-it(E_n-E_m)}$: erratic oscillations in Time

Very late Times: oscillations average out and $n=m$ dominates

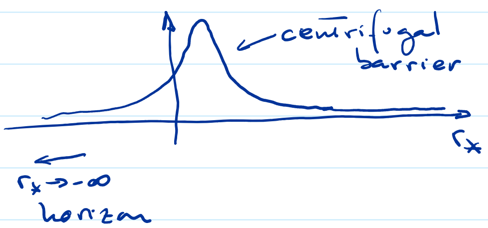
$$\langle O(t)\, O(0)\rangle_\beta \longrightarrow \sum_n e^{-\beta E_n} \left|\langle n|O|n\rangle\right|^2$$

Correlations initially decay, but never completely disappear:

This does not happen in a black hole.

Initial decay $e^{-t/\tau}$: quasinormal decay

Massless field propagation in bh:

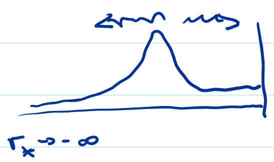


$$-\frac{\partial^2 \psi}{\partial r_*^2} = (\omega^2 - V)\,\psi(r_*)$$

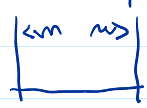
Study The energy spectrum of bh oscillations

Put black hole in a box at $r_* = R$.

Spectrum is not discrete, but continuous and with absorptive part



Normal spectrum



Absorption by horizon

$r_* \to -\infty$

Absorption by horizon

Quasinormal spectrum $\omega = \omega_0 - i/\tau$

$$e^{-i\omega t} = e^{-i\omega_0 t - t/\tau}$$

decc

In a black hole

$$\langle \mathcal{O}(t)\, \mathcal{O}(0) \rangle_\beta \sim e^{-t/\tau} \xrightarrow{t\to\infty} 0$$

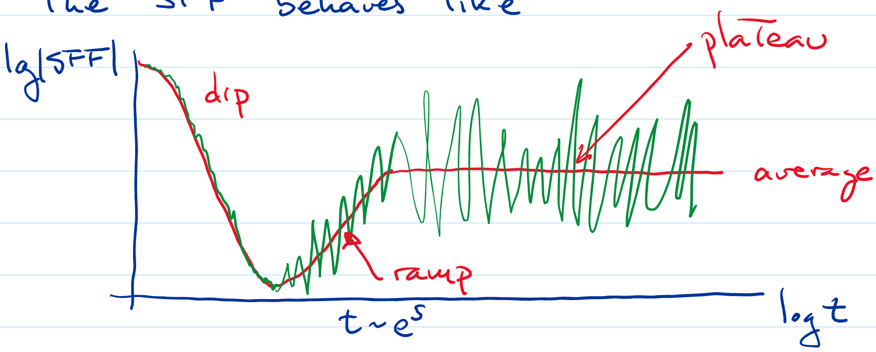
Correlations disappear completely.
Black hole in box does not behave according to QM. Info about initial perturbation is <u>completely lost</u>.

Simpler observable: <u>Spectral form factor</u>

$|\langle n|\mathcal{O}|m\rangle|^2$ is generally expected (ETH) to be a smooth function of energy.

$$\sum_{n,m} e^{-(\beta+it)E_n} e^{-(\beta-it)E_m} = \underbrace{Z(\beta+it)\, Z(\beta-it)}_{\text{"Spectral form factor"}}$$

In a chaotic system with a random energy spectrum (eigenvalue statistics given by random matrix distribution)
The SFF behaves like



dip: Thermal relaxation

ramp: eigenvalue repulsion

plateau: discreteness of spectrum

ramp: eigenvalue repulsion

plateau: discreteness of spectrum

The GPI w/ wormholes allows to reproduce

the ramp.