# Exercises 3

August 2024

# 1 Holographic 4-pt functions: how to succeed without really trying

*Consider an interacting scalar quadruplet $\phi_i$ ($i = 1, 2, 3, 4$) of masses $m_i$ and confomal dimensions $\Delta_i$ in Euclidean $AdS_{D+1}$. Starting from the GPKW prescription, try to write down the expression for 4-pt functions exploiting the isometries of $AdS_{D+1}$ to avoid explicit calculation of Witten diagrams.*

## 1.1 Definition of the amplitudes

We work in the Euclidean metric

$$ds^2 = \frac{1}{z_0^2}\left(dz_0^2 + d\mathbf{z}^2\right), \quad z = (z_0, \mathbf{z}) = (z_0, z_1, \dots z_D). \tag{1}$$

The bulk-to-bulk propagator $G_\Delta$ is defined to obey the wave equation

$$(-\nabla^2 + m^2)G_\Delta(\xi) = \frac{\delta(z-w)}{\sqrt{g}}, \qquad \xi = \frac{(z-w)^2}{2z_0 w_0} \tag{2}$$

In $\mathrm{AdS}_{D+1}$, this turns out to be solved by

$$G_\Delta(\xi) \sim \xi^{-\Delta}{}_2F_1\left(\Delta, \Delta - \frac{D}{2} + \frac{1}{2}, 2\Delta - D + 1; \frac{1}{\xi^2}\right). \tag{3}$$

We will assume that both $\Delta, \Delta_i \geq \frac{D}{2}$. The bulk-to-boundary propagator from a bulk point $(z, \mathbf{z})$ to a boundary point $(0, \mathbf{z}')$ is given by

$$\mathcal{G}_\Delta(z_0, \mathbf{z}, \mathbf{z}') = \left(\frac{z_0}{z_0^2 + |\mathbf{z} - \mathbf{z}'|^2}\right)^\Delta. \tag{4}$$

According to GPKW formula, in order to compute a 4-pt function we need to compute the Witten diagram from Fig. 1. Explicitly, the amplitude reads ($i = 1, 2, 3, 4$):

$$A_4(\mathbf{x}_i) = \int \frac{d^{D+1}w}{w_0^{D+1}} A_3\left(w, \mathbf{x}_1, \mathbf{x}_3\right) \mathcal{G}_{\Delta_2}(w, \mathbf{x}_2)\mathcal{G}_{\Delta_4}(w, \mathbf{x}_4), \tag{5}$$

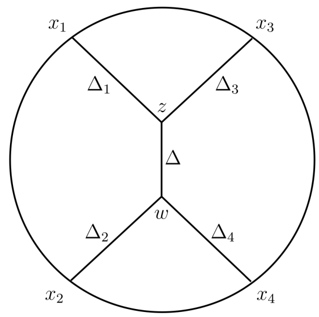


Figure 1: Witten diagram associated to an AdS four-point function

with

$$A_3(w, \mathbf{x}_1, \mathbf{x}_3) = \int \frac{d^{D+1}u}{u_0^{D+1}} G_\Delta(\xi) \mathcal{G}_{\Delta_1}(u, \mathbf{x}_1) \mathcal{G}_{\Delta_3}(u, \mathbf{x}_3). \tag{6}$$

1. Write the functions $A_3$, $\mathcal{G}$ and $G$ that appear in (5) and (6) next to the corresponding lines/points in the Witten diagram.

(5) is a hard integral to explicitly perform. The neat trick proposed in [1] is to exploit the fact that AdS isometries become conformal isometries in dual CFT. We will use it to kill the integration in (6) coming from the Witten diagrams. We cannot kill the integral in (5) but doing one instead of two is still a big advantage.

## 1.2 Properties of the inversion

The integral (6) can be simplified significantly by performing the inversion transformation on the coordinates:

$$x^\mu \mapsto \bar{x}^\mu \equiv \frac{x^\mu}{x^2}, \tag{7}$$

Let's see how the fields and the integration measure transform under (7).

2. Show that

$$\mathcal{G}_\Delta(\bar{z}, \bar{\mathbf{x}}, \bar{\mathbf{x}'}) = \mathcal{G}_\Delta(z, \mathbf{x}, \mathbf{x}')|\mathbf{x}'|^{2\Delta}. \tag{8}$$

## 1.3 Transforming the integral

Start by exploiting the translation invariance to move $\mathbf{x}_1 \to 0$, $\mathbf{x}_3 \to \mathbf{x}_{31} \equiv \mathbf{x}_3 - \mathbf{x}_1$. Apply then the inversion transformation to the integration variables $w, u$ and also to $\mathbf{x}_{31}$ (or if you want to be pedantic to the bulk point $(0, \mathbf{x}_{31})$):

$$(w_\mu, u_\mu, \mathbf{x}_{31}) \mapsto (w'_\mu, u'_\mu, \mathbf{x}'_{31}) = \left( \frac{w_\mu}{w^2}, \frac{u_\mu}{u^2}, \frac{\mathbf{x}_{31}}{x_{13}^2} \right). \tag{9}$$

3. Show that the integral (6) becomes

$$A_3(w, \mathbf{x}_1, \mathbf{x}_3) = |\mathbf{x}_{31}|^{-2\Delta_3} I(w' - x'_{31}), \tag{10}$$

where

$$I(w) = \int \frac{d^{D+1}u}{u_0^{D+1}} G_\Delta(\xi) u_0^{\Delta_1} \left(\frac{u_0}{u^2}\right)^{\Delta_3}. \tag{11}$$

You can use the fact that

$$\frac{d^{D+1}\bar{z}}{\bar{z_0}^{D+1}} = \frac{d^{D+1}z}{z_0^{D+1}}. \tag{12}$$

Now we turn the crank. The integral $I(w)$ is invariant to rescaling $w \mapsto \lambda w$. This is because the chord distance $\xi$ is scale-invariant, and that's the only place where $w$ appears in $I$. The integral is invariant to all Poincaré transformations on the boundary. This is because the geodesic distance $\xi$ is invariant under these, if we transform $z$ simultaneously. The module $u^2$ is unchanged by Poincare isometries and $u_0$ does not see them at all.

4. Show that the geodesic distance $\xi$ is invariant under Poincaré transformations on the boundary. Explain why this means that $I(w)$ is also invariant under these transformations.

5. Show that under $w_\mu \to \lambda w_\mu$,

$$I(\lambda w) = \lambda^{\Delta_1 - \Delta_3} I(w). \tag{13}$$

6. Explain that this means that the function $I(w)$ is constrained to be of the form

$$I(w) = w_0^{\Delta_{13}} f\left(\frac{w_0^2}{w^2}\right) = w_0^{\Delta_{13}} f\left(\frac{w_0^2}{w_0^2 + |\mathbf{w}|^2}\right), \quad \Delta_{13} \equiv \Delta_1 - \Delta_3. \tag{14}$$

7. We want to solve for $f(s)$, where we write $s = \frac{w_0^2}{w^2}$. We can evaluate the free part of the equation of motion on $I(w)$ from (11). Show that

$$\left(-\nabla^2 + m^2\right) I(w) = w_0^{\Delta_{13}} \left(\frac{w_0^2}{w^2}\right)^{\Delta_3}. \tag{15}$$

8. Now show that $(-\nabla^2 + m^2)I(w)$ is also equal to

$$4w_0^{\Delta_{13}} s^2(s-1)f'' + 4sw_0^{\Delta_{13}} \left[(\Delta_{13}+1)s - \Delta_{13} + \frac{D}{2} - 1\right] f' + w_0^{\Delta_{13}} \left[\Delta_{13}(D - \Delta_{13}) + m^2\right] f. \tag{16}$$

Now, equating (15) and (16) yields an inhomogeneous differential equation for $f$. We can impose some boundary conditions, based on smoothness conditions and asymptotics.

9. Explain why you expect $f$ to be smooth as $s \to 1$.

10. Show that when $w_0 \to 0$, one should have $I \sim w_0^\Delta$. Also show that this means that $f$ should behave under this limit as

$$f(s \to 0) \sim s^{(\Delta-\Delta_{13})/2}. \tag{17}$$

11. It is convenient to look for $f$ in a series representation, as $f = \sum_l a_l s^{\Delta_3+l}$. Substituting into the differential equation (15)=(16), show that we get

$$\begin{aligned} a_l &= 0, & l \geq 0 \\ a_{-1} &= \frac{1}{4(\Delta_1-1)(\Delta_3-1)} \\ a_l &= \frac{\left(l+\frac{\Delta_1+\Delta_3-\Delta}{2}\right)\left(l+\frac{\Delta_1+\Delta_3+\Delta-D}{2}\right)}{(\Delta_3+l-1)(\Delta_1+l-1)} a_{l+1}, & l < -1. \end{aligned} \tag{18}$$

12. Under what condition for $\Delta$ and $\Delta_i$ does this sum terminate? This is always satisfied for supersymmetric gauge theories.

Now that we have all this, we can insert the solution for $I(w)$ into $A_3$ from (10) – we didn't do any integrations, just a few isometries and a recurrent algebraic equation for the coefficients $a_l$. We still need to do the integral in $A_4$.

13. Show that this becomes

$$A_4 = \sum_l a_l |\mathbf{x}_{13}|^{2l} \int \frac{d^{D+1}w}{w_0^{D+1}} \mathcal{G}_{\Delta_1+l}(w,\mathbf{x}_1)\mathcal{G}_{\Delta_3+l}(w,\mathbf{x}_3)\mathcal{G}_{\Delta_2}(w,\mathbf{x}_2)\mathcal{G}_{\Delta_4}(w,\mathbf{x}_4). \tag{19}$$

This is a finite sum of single integrals, which is a lot easier to deal with than the integral we started with!

# 2 3D Gravity through group theory

We will show in this exercise how we can understand $\mathrm{AdS}_3$ in terms of group theory. Two ways of writing the $\mathrm{AdS}_3$ metric are

$$\begin{aligned} ds^2 &= -dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2 \\ &= -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\phi^2. \end{aligned} \tag{20}$$

1. Start by showing that

$$\begin{aligned} g &= e^{\frac{i}{2}(t+\phi)\sigma_2} e^{\rho\sigma_3} e^{\frac{i}{2}(t-\phi)\sigma_2} \\ &= \begin{pmatrix} \cos t\cosh\rho + \cos\phi\sinh\rho & \sin t\cosh\rho - \sin\phi\sinh\rho \\ -\sin t\cosh\rho - \sin\phi\sinh\rho & \cos t\cosh\rho - \cos\phi\sinh\rho \end{pmatrix}, \end{aligned} \tag{21}$$

Here $\sigma_i$ are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{22}$$

2. Check that $\det g = 1$. What group is parametrized by $g$?

3. Compute the object (hint: use Mathematica)

$$\frac{1}{2}\operatorname{Tr}((g^{-1}dg))^2. \tag{23}$$

   Observe that it is equal to the $\mathrm{AdS}_3$ metric!

Now we try to do the same for the BTZ black hole!

4. Now show that

$$\begin{aligned} g &= e^{\varphi\sigma^3} e^{\rho\sigma^1} e^{\psi\sigma^3} \\ &= \begin{pmatrix} e^{\varphi} & 0 \\ 0 & e^{-\varphi} \end{pmatrix} \begin{pmatrix} r & \sqrt{r^2-1} \\ \sqrt{r^2-1} & r \end{pmatrix} \begin{pmatrix} e^{\psi} & 0 \\ 0 & e^{-\psi} \end{pmatrix}. \end{aligned} \tag{24}$$

   Here $r = \cosh\rho$. Show again that $\det g = 1$.

5. Now compute the metric from here in terms of $(t, \phi, r)$ when $\phi = \varphi + \psi$ and $t = \varphi - \psi$. Show that it reduces to

$$ds^2 = -(r^2-1)dt^2 + \frac{dr^2}{r^2-1} + r^2 d\phi^2. \tag{25}$$

We have seen that we can rephrase $\mathrm{AdS}_3$ gravity in terms of $\mathrm{SL}(2,\mathbb{R})$ group theory!

# References

[1] Eric D'Hoker, Daniel Z. Freedman, and Leonardo Rastelli. "AdS/CFT four point functions: How to succeed at z integrals without really trying". In: *Nucl. Phys. B* 562 (1999), pp. 395–411. DOI: 10.1016/S0550-3213(99)00526-X. arXiv: hep-th/9905049.