Exercises 4

August 2024

1 Black hole information paradox

In this exercise session, you will go through some of the details left open during the lecture of the computation done in `https://arxiv.org/pdf/1905.08762`.

1. For the four-dimensional Reissner Nordström black hole, the metric is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \qquad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{1}$$

   Show how for near extremal black holes, with $M = Q + \Delta M$ with ΔM small, the metric reduces to the metric for a black hole in $\mathrm{AdS}_2 \times S^2$:

$$Q^{-2} ds^2 = -(\tilde{r}^2 - r_h^2)dt^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 - r_h^2} + d\Omega_2^2. \tag{2}$$

2. The AdS metric in Poincaré coordinates is given by

$$ds^2 = \frac{-dt^2 + dz^2}{z^2}, \tag{3}$$

   and the dilaton profile is given by

$$\phi = 2\bar{\phi}_r \frac{1 - (\pi T_0)^2 x^+ x^-}{x^+ - x^-}. \tag{4}$$

   We cut off the AdS metric by some boundary with proper time u, which is related to the AdS asymptotic time t through $t = f(u)$. Imposing

$$g_{uu}|_{\mathrm{bdy}} = \frac{1}{\epsilon^2}, \qquad \phi_{\mathrm{bdy}} = \frac{\bar{\phi}_r}{\epsilon} \tag{5}$$

   for some small ϵ, solve for $t = f(u)$ and $z(u)$.

3. In Euclidean signature, we write

$$ds^2 = \frac{4dxd\bar{x}}{(x + \bar{x})^2} \tag{6}$$

where $x = z - t$ and $\bar{x} = z + t$. We have cut off the spacetime by the boundary particle, parameterized by $t = f(u)$ and $z = \epsilon f'(u)$. Writing $x = f(y)$ and $\bar{x} = f(\bar{y})$, the AdS metric is conformally flat:

$$ds^2 = \Omega_{y,\bar{y}}^2 dy d\bar{y}. \tag{7}$$

Show that in these $y, \bar{y}$ coordinates, the location of the boundary is given by $y + \bar{y} = 2\epsilon$ and at this boundary, $\frac{\bar{y}-y}{2} = u$. This means that we can glue the AdS spacetime to the flat bath in a nice way. You can assume in this problem that $f(-y) = -f(y)$.

4. The Schwarzian derivative is defined as

$$\{w, x\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \tag{8}$$

The Möbius transformations are defined through

$$w(x) = \frac{ax + b}{cx + d}. \tag{9}$$

Show that for these types of transformations, the Schwarzian derivative is zero.

5. We computed some Rényi entropies in 2D BCFT. We used the formula

$$S^{(n)} = -\frac{1}{n-1}\log\langle\sigma(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\rangle, \quad \Delta_n = \frac{c}{12}\frac{(n-1)(n+1)}{n}. \tag{10}$$

where the two-point functions in BCFT are equal to four-point functions in CFT with half the conformal dimension. From the behaviour of primaries under conformal dimensions, prove that

$$S_{\Omega^{-2}g} = S_g - \frac{c}{6}\sum_{\text{endpoints}} \log\Omega. \tag{11}$$

6. Show that the conformal cross-ratio

$$\eta = \frac{(w_1 - \bar{w}_1)(w_2 - \bar{w}_2)}{(w_1 - \bar{w}_2)(w_2 - \bar{w}_1)} \tag{12}$$

is invariant under conformal transformations.

The entropy of an interval in flat half-space CFT, including the boundary, is given by

$$S = \frac{c}{6}\log(w + \bar{w}) + \log g. \tag{13}$$

Having an interval not including the boundary, we have

$$S = \frac{c}{6}\log((w_1 - w_2)(\bar{w}_1 - \bar{w}_2)\eta) + \log G(\eta). \tag{14}$$

7. Now we will compute some entropies in the AdS coordinates. We will first do some warm-up exercise. Consider an interval with *both* endpoints to the past of the shock: $x^+_{1,2} > 0$ and $x^-_{1,2} < 0$. We take one of the endpoints all the way into the AdS, the other is free. Show that the entropy of such an interval (in AdS coordinates) is given by

$$S = \frac{c}{6}\log 2 + \log g. \tag{15}$$

8. Now we move on to the more complicated intervals. First study the case with one endpoint to the future (x_1) and one end point to the past (x_2). In the $w_0 \to 0$ limit, show that

$$S = \frac{c}{6}\log\left[\frac{48\pi E_S}{c}\frac{-y_1^- x_1^+ x_2^- (x_2^+ - x_1^+)\sqrt{f'(y_1^-)}}{x_2^+(x_1^+ - x_1^-)(x_1^+ - x_2^-)}\right] + \log G(\eta). \tag{16}$$

Hint: you might first want to show that

$$\eta = \frac{x_1^+(x_2^+ - x_2^-)}{x_2^+(x_1^+ - x_2^-)}. \tag{17}$$

9. Lastly, take both endpoints to the future of the shock. First show that in this case, $\eta \approx 1$. This limit corresponds to the OPE limit, where we take both coordinates to lie very close to each other. This means that we can approximate the two-point BCFT function as a CFT, which means that $G(1) \approx 1$. Show that now in this case, again in the $w_0 \to 0$ limit,

$$S = \frac{c}{6}\log\left[\frac{4(y_1^- - y_2^-)(x_2^+ - x_1^+)\sqrt{f'(y_1^-)f'(y_2^-)}}{(x_1^+ - x_1^-)(x_2^+ - x_2^-)}\right] \tag{18}$$