Exercises 2

August 2024

1 Rényi entropies and von Neumann entropy

GHC and W state. In this exercise, we will compute some Rényi entropies and von Neumann entropies, and prove an interesting relation between the two. Recall that for a density matrix ρ, the von Neumann entropy is defined as $S(\rho) = -\operatorname{Tr}\rho\log\rho$. The n'th Rényi entropy is defined as

$$S^{(n)}(\rho) = \frac{1}{1-n}\log\operatorname{Tr}\rho^n, \qquad n \geq 2. \tag{1}$$

1. The GHZ state is a 3-qubit state defined as

$$|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}. \tag{2}$$

   Compute both the von Neumann entropy and the Rényi entropy (for general $n \geq 2$) for ρ_1 and ρ_{12}.

2. Repeat the same calculations for the W-state:

$$|W\rangle = \frac{|100\rangle + |010\rangle + |001\rangle}{\sqrt{3}}. \tag{3}$$

3. If you analytically continue the n'th Rényi entropy to be some function of *all* n (not only integers with $n \geq 2$), then one can compute the limit of the Rényi entropy with n going to 1. Compute this, for the Rényi entropies you computed in the previous exercises.

4. If you did everything correctly, you should have found that the limit reduces to the von Neumann entropy:

$$\lim_{n\to 1} S^{(n)}(\rho) = S(\rho). \tag{4}$$

   Now we are going to prove it, for general density matrices! In terms of the eigenvalues λ_i of some density matrix ρ, the von Neumann entropy is given by

$$S(\rho) = -\sum_i \lambda_i \log\lambda_i. \tag{5}$$

We are going to show that the limit of the Rényi entropies obeys the same formula. First, show that

$$\operatorname{Tr}\rho^n = 1 + \sum_i \lambda_i(\lambda_i^{n-1} - 1). \tag{6}$$

5. Then, show that in the limit of n going to 1,

$$\lim_{n\to 1} \frac{1}{1-n} \log \operatorname{Tr}\rho^n = \lim_{n\to 1} \frac{1}{1-n} \sum_i \lambda_i(\lambda_i^{n-1} - 1). \tag{7}$$

   From here, prove (4).

6. *Optional* Another way of obtaining the von Neumann entropy from the Rényi entropies is

$$S(\rho) = -\frac{d}{dn} \operatorname{Tr}\rho^n \bigg|_{n=1}. \tag{8}$$

   Prove that this is indeed the case, for general density matrices.

2 CFT Two-point function

Euclidean correlation functions are always time ordered:

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = \langle 0 | T\{\hat{\mathcal{O}}_1(\tau_1, \mathbf{x}_1) \dots \hat{\mathcal{O}}_n(\tau_n, \mathbf{x}_n)\} | 0 \rangle . \tag{9}$$

The time evolution is generated by the Hamiltonian $\hat{H}$:

$$\hat{\mathcal{O}}(\tau, \mathbf{x}) = e^{\tau \hat{H}} \hat{\mathcal{O}}(0, \mathbf{x}) e^{-\tau \hat{H}}. \tag{10}$$

1. Consider the correlation function

$$\langle 0 | \hat{O}(\tau_1, \mathbf{x}_1) \hat{O}(\tau_2, \mathbf{x}_2) | 0 \rangle . \tag{11}$$

   Show that, if the operators are anti-time ordered ($\tau_2 > \tau_1$), that this correlator is infinite, whereas for time-ordered correlators it is bounded.

2. From conformal symmetry, we know

$$\langle \mathcal{O}(\tau, \mathbf{x}) \mathcal{O}(0) \rangle = \frac{1}{(\tau^2 + \mathbf{x}^2)^\Delta}. \tag{12}$$

   Wick rotating to Lorentzian, where are the singularities for generic Δ?

3. Writing $\tau = it + \epsilon$ with $\epsilon \to 0^+$, write down the Lorentzian correlator. Show how the correlator obtains a phase, dependant on the causal configuration.