Solutions PSI 2024

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1 Exercise session 1

1.1 Black hole thermodynamics from Euclidean Quantum Gravity

1.1.1 Schwarzschild black hole thermodynamics

1. Fixing the length of the Euclidean time circle, we can relate β_0 to β as

$$\int_0^\beta \sqrt{h_{\tau\tau}} = \beta\sqrt{1-\frac{2GM}{R_b}} \approx \beta\left(1-\frac{GM}{R_b}\right) = \int_0^{\beta_0} \sqrt{h^0_{\tau\tau}} = \beta_0. \tag{1}$$

2. Now we want to compute

$$-\frac{1}{8\pi G}\int \sqrt{h}K. \tag{2}$$

We can use $\sqrt{h}K = n^\mu \partial_\mu \sqrt{h}$. We have

$$n^\mu n_\mu = 1 = n^r n^r g_{rr} \longrightarrow n^r = \frac{1}{\sqrt{g_{rr}}} = \sqrt{1-\frac{2GM}{R_b}} \approx 1-\frac{GM}{R_b}. \tag{3}$$

We will also use

$$\sqrt{h} = \sqrt{1-\frac{2GM}{r}}r^2 \approx r^2\left(1-\frac{GM}{r}\right)|_{r=R_b}. \tag{4}$$

We can take a derivative with respect to r to find

$$\partial_r\sqrt{h} = 2R_b - GM. \tag{5}$$

So we have, for the Schwarzschild spacetime (using $\int d\Omega_2 = 4\pi$),

$$\begin{aligned}-\frac{1}{8\pi G}\int\sqrt{h}K &\approx -\frac{4\pi\beta}{8\pi G}\left(1-\frac{GM}{R_b}\right)(2R_b-GM)\\ &\approx -\frac{\beta}{G}R_b\left(1-\frac{3GM}{2R_b}\right).\end{aligned} \tag{6}$$

For the flat Minkowski metric, we can use the same result, replacing $\beta \to \beta_0$ and $M \to 0$. So for the flat spacetime:

$$-\frac{1}{8\pi G}\int\sqrt{h}K \approx -\frac{\beta}{G}R_b\left(1-\frac{GM}{R_b}\right). \tag{7}$$

Subtracting this, we find

$$I_E[g^{(\mathrm{cl})}] - I_E[g^0] = \frac{\beta M}{2}. \tag{8}$$

3. For the Schwarzschild black hole, we have $\beta = 8\pi GM$. So

$$\beta F = \frac{\beta^2}{16\pi G}. \tag{9}$$

Then

$$E = \partial_\beta \beta F = \frac{\beta}{8\pi G} = M. \tag{10}$$

We can also compute

$$S = (\beta\partial_\beta - 1)\beta F = \frac{\beta^2}{16\pi G}. \tag{11}$$

The area of a Schwarzschild black hole is $4\pi r_h^2 = 4\pi(2GM)^2 = 16\pi G^2 M^2 = 16\pi G^2 \frac{\beta^2}{(8\pi G)^2} = \frac{\beta^2}{4\pi}$. So we see that indeed, $S = \frac{A}{4G}$.

1.1.2 AdS black brane

The metric of the five-dimensional black brane is given by

$$ds^2 = \frac{r^2}{\ell^2}\left(f(r)d\tau^2 + \sum_{i=1}^{3} dx_i^2\right) + \frac{\ell^2}{r^2}\frac{dr^2}{r^2}, \qquad f(r) = 1 - \frac{r_H^4}{4}. \tag{12}$$

1. The temperature can be found by expanding near the horizon, and imposing that it locally looks like flat space there. Writing $r = r_H + \xi$, we obtain

$$f(r) \approx 1 - \frac{r_H^4}{r_H^4 + 4r_H^3\xi} \approx \frac{4\xi}{r_H}. \tag{13}$$

This yields the near-horizon metric

$$ds^2 = \frac{r_H^2}{\ell^2}\left(\frac{4\xi}{r_H}d\tau^2 + \sum_{i=1}^{3} dx_i^2\right) + \frac{\ell^2}{r_H^2}\frac{r_H d\xi^2}{4\xi}. \tag{14}$$

This is not in the form we want yet. In particular, we want a radial part $\sim d\tilde{r}^2$. So we impose

$$d\tilde{r} = \frac{\ell}{\sqrt{r_H}}\frac{1}{2\sqrt{\xi}}d\xi \longrightarrow \tilde{r} = \frac{\ell}{\sqrt{r_H}}\sqrt{\xi}. \tag{15}$$

Then the metric becomes

$$\frac{4r_H^2\tilde{r}^2}{\ell^4}d\tau^2 + \frac{r_H^2}{\ell^2}\sum_{i=1}^{3} dx_i^2 + d\tilde{r}^2. \tag{16}$$

To avoid a conical singularity, we must thus impose

$$2r_H/\ell^2\tau \sim 2r_H/\ell^2\tau + 2\pi \quad \longrightarrow \quad \tau \sim \tau + \frac{\pi\ell^2}{r_H} \tag{17}$$

so we read off

$$\beta = \frac{\pi\ell^2}{r_H}. \tag{18}$$

2. We first match the Euclidean time circles again:

$$\beta_0 = \beta\sqrt{1-\frac{r_H^4}{R_b^4}} \approx \beta\left(1-\frac{r_H^4}{2R_b^4}\right). \tag{19}$$

Now the bulk term does not vanish. We have

$$R_{\mu\nu} = -\frac{4}{\ell^2}g_{\mu\nu} \quad\longrightarrow\quad R = g^{\mu\nu}R_{\mu\nu} = -\frac{4}{\ell^2}g^\mu_\mu = -\frac{20}{\ell^2}. \tag{20}$$

So we see $R+\frac{12}{\ell^2} = -\frac{8}{\ell^2}$. The integral over x_1, x_2, x_3 yields V, but since we divide by V, it becomes 1. The integral over r goes from r_H to R_b. Lastly, we use that

$$\sqrt{-g} = \left(\frac{r}{\ell}\right)^3. \tag{21}$$

So we obtain

$$-\frac{1}{V}\frac{1}{16\pi G}\int_{\mathcal{M}}\frac{r^3}{\ell^3} - \cdot\frac{8}{\ell^2} = \frac{\beta}{16\pi G}\frac{8}{\ell^2}\frac{1}{\ell^3}\frac{r^4}{4}\bigg|_{r_H}^{R_b} = \frac{\beta}{8\pi G\ell^5}R_b^4\left(1-\frac{r_H^4}{R_b^4}\right). \tag{22}$$

For the empty solution, we must set $r_H = 0$ and take β_0 instead of β, so we obtain

$$\frac{\beta}{8\pi G\ell^5}R_b^4\left(1-\frac{r_H^4}{2R_b^4}\right). \tag{23}$$

Subtracting both solutions gives

$$-\frac{\beta}{16\pi G\ell}\frac{r_H^4}{\ell^4}. \tag{24}$$

Now we compute the bulk contribution. We have

$$n^r = \frac{1}{\sqrt{g_{rr}}} = \sqrt{\frac{r^2}{\ell^2}f(r)} \approx \frac{r}{\ell}\left(1-\frac{r_H^4}{2r^4}\right), \tag{25}$$

using the fact that it will be evaluated at large r. We also have

$$\sqrt{h} = \sqrt{\frac{r^8}{\ell^8}f(r)} \approx \frac{r^4}{\ell^4}\left(1-\frac{r_H^4}{2r^4}\right). \tag{26}$$

Then

$$\sqrt{h}K = n^r\partial_r\sqrt{h} \approx \frac{R_b}{\ell}\left(1-\frac{r_H^4}{2R_b^4}\right)\frac{4R_b^3}{\ell^4}. \tag{27}$$

So

$$-\frac{1}{V}\frac{1}{8\pi G}\int_{\partial\mathcal{M}}\sqrt{h}K \approx -\frac{\beta}{2\pi G\ell}\frac{R_b^4}{\ell^4}\left(1-\frac{r_H^4}{2R_b^4}\right). \tag{28}$$

The same term for the empty AdS space sets $r_H = 0$ in the above computation, and sets $\beta \to \beta_0$. But we see that we will get exactly the same as above! Thus to leading order, the boundary term cancels between the AdS black brane and the empty AdS space. We thus have

$$\beta f = -\frac{\beta}{16\pi G\ell}\frac{r_H^4}{\ell^4}. \tag{29}$$

We now try to rewrite this in terms of β. Using $\beta = \frac{\pi\ell^2}{r_H}$, we see

$$f = -\frac{1}{16\pi G\ell}\frac{\ell^4\pi^4}{\beta^4} = -\frac{\pi^3\ell^3}{16G}\frac{1}{\beta^4}. \tag{30}$$

3. Computing the energy density,

$$e = \frac{E}{V} = \partial_\beta(\beta f) = \frac{3\pi^3\ell^3}{16G}\frac{1}{\beta^4}. \tag{31}$$

The entropy density is given by

$$s = \frac{S}{V} = (\beta\partial_\beta - 1)(\beta f) = \frac{\pi^3\ell^3}{4G}\frac{1}{\beta^3} = \frac{r_H^3}{4G\ell^3}. \tag{32}$$

The area density is indeed $a = A/V = r_H^3/\ell^3$, so we see that $s = a/4G$ as expected.

4. The specific heat is given by

$$c = T\frac{\partial s}{dT} = -\beta\partial_\beta s = \frac{3\pi\ell^3}{4G}\frac{1}{\beta^3}. \tag{33}$$

This is positive. This tells us that AdS black branes are stable, and do generically not evaporate.

1.1.3 AdS black hole

The Euclidean metric for the five-dimensional black hole is given by

$$ds^2 = \left(1 - \frac{\mu}{r^2} + \frac{r^2}{\ell^2}\right)d\tau^2 + \frac{dr^2}{1 - \frac{\mu}{r^2} + \frac{r^2}{\ell^2}} + r^2 d\Omega_3^2. \tag{34}$$

One can solve for r_+ in terms of μ and finds

$$\mu = \frac{r_+^2(r_+^2 + \ell^2)}{\ell^2}. \tag{35}$$

1. We begin again by expanding in the near-horizon limit. Writing $r = r_+ + \xi$,

$$\begin{aligned} f(r) &\approx 1 - \frac{\mu}{r_+^2 + 2r_+\xi} + \frac{r_+^2 + 2r_+\xi}{\ell^2} \approx 1 - \frac{\mu}{r_+^2}\left(1 - 2\frac{\xi}{r_+}\right) + \frac{r_+^2 + 2r_+\xi}{\ell^2} \\ &= \frac{2\mu\xi}{r_+^3} + \frac{2r_+\xi}{\ell^2} = 2\frac{\xi(2r_+^2 + \ell^2)}{\ell^2 r_+} \end{aligned} \tag{36}$$

The r-part of the metric then becomes

$$\frac{d\xi^2}{\xi}\frac{\ell^2 r_+}{2(2r_+^2 + \ell^2)}. \tag{37}$$

We now write

$$\tilde{r} = \sqrt{2\xi r_+}\frac{\ell}{\sqrt{2r_+^2 + \ell^2}}. \tag{38}$$

Then

$$\frac{2\xi(2r_+^2 + \ell^2)}{\ell^2 r_+}d\tau^2 = \tilde{r}^2\frac{(2r_+^2 + \ell^2)^2}{\ell^4 r_+^2}d\tau^2. \tag{39}$$

From here we read off that

$$\beta = \frac{2\pi\ell^2 r_+}{2r_+^2 + \ell^2}. \tag{40}$$

2. Fixing the size of the Euclidean time circle yields

$$\begin{aligned}\beta_0\sqrt{1+\frac{R_b^2}{\ell^2}} &= \beta\sqrt{1-\frac{\mu}{R_b^2}+\frac{R_b^2}{\ell^2}} = \beta\sqrt{1+\frac{R_b^2}{\ell^2}}\sqrt{1-\frac{\mu}{R_b^2}\frac{1}{1+\frac{R_b^2}{\ell^2}}} \\ \beta_0 &\approx \beta\sqrt{1-\frac{\mu\ell^2}{2R_b^4}} \approx \beta\left(1-\frac{\mu\ell^2}{2R_b^4}\right).\end{aligned} \tag{41}$$

Noting that the integral over S^3 yields $2\pi^2$, we obtain

$$\begin{aligned}-\frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{g}\left(R+\frac{12}{\ell^2}\right) &= \frac{2\pi^2\beta}{16\pi G}\frac{8}{\ell^2}\int_{r_+}^{R_b} r^3 dr \\ &= \frac{\beta\pi}{4G\ell^2}R_b^4\left(1-\frac{r_+^4}{R_b^4}\right).\end{aligned} \tag{42}$$

The same term for the empty AdS gives

$$\begin{aligned}-\frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{g^0}\left(R+\frac{12}{\ell^2}\right) &= \frac{2\pi^2\beta_0}{16\pi G}\frac{8}{\ell^2}\int_0^{R_b} r^3 dr \\ &= \frac{\beta\pi}{4G\ell^2}R_b^4\left(1-\frac{\mu\ell^2}{2R_b^4}\right) \\ &= \frac{\beta\pi}{4G\ell^2}R_b^4\left(1-\frac{r_+^2(r_+^2+\ell^2)}{2R_b^4}\right)\end{aligned} \tag{43}$$

Subtracting this from the black hole solution gives for the bulk terms:

$$-\frac{\beta\pi}{4\pi G\ell^2}\frac{r_+^2(r_+^2-\ell^2)}{2}. \tag{44}$$

Now we move on to the boundary terms again. Note that

$$n^r = \frac{1}{\sqrt{g_{rr}}} = \sqrt{1-\frac{\mu}{R_b^2}+\frac{R_b^2}{\ell^2}} \approx \frac{R_b}{\ell}\left(1-\frac{\mu\ell^2}{2R_b^4}\right). \tag{45}$$

We also see

$$\sqrt{h} = r^3\sqrt{1-\frac{\mu}{r^2}+\frac{r^2}{\ell^2}} \approx \frac{r^4}{\ell}\left(1-\frac{\mu\ell^2}{2r^4}\right). \tag{46}$$

Then

$$\sqrt{h}K \approx \frac{R_b}{\ell}\left(1-\frac{\mu\ell^2}{2R_b^4}\right)\frac{4R_b^3}{\ell} = \frac{4R_b^4}{\ell^2}\left(1-\frac{\mu\ell^2}{2R_b^4}\right). \tag{47}$$

So we have

$$-\frac{1}{8\pi G}\int_{\partial\mathcal{M}}\sqrt{h}K \approx \frac{2\pi^2\beta}{8\pi G}\frac{4R_b^4}{\ell^2}\left(1-\frac{\mu\ell^2}{2R_b^4}\right). \tag{48}$$

For empty AdS, we set $\mu = 0$ and replace β by β_0. This gives exactly the same term (to this order in $1/R_b$), so we find

$$F \approx -\frac{\pi}{8\pi G}\frac{r_+^2}{\ell^2}(r_+^2-\ell^2). \tag{49}$$

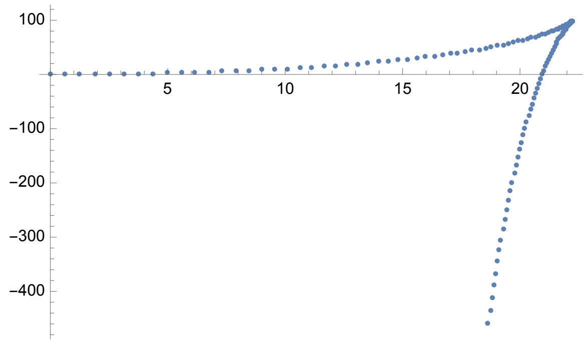

Figure 1: Phase diagram of F plotted as a function of β

3. r_+ is related to β in some complicated way. It will therefore not be straightforward to rewrite F in terms of β. It is much easier to plug the expressions into Mathematica. Then you will find
$$E = \partial_\beta \beta F = \frac{3\pi}{8G}\frac{r_+^2(r_+^2+\ell^2)}{\ell^2} = \frac{3\pi}{8G}\mu. \tag{50}$$

4. The specific heat can be computed once we have the entropy. Mathematica will give us
$$S = \frac{\pi^2 r_+^3}{2G}. \tag{51}$$
The specific heat then is given by
$$C = -\beta\partial_\beta S = \frac{3\pi^2 r_+^3(2r_+^2+\ell^2)}{2G(2r_+^2-\ell^2)}. \tag{52}$$
We thus see that this is positive if $r_+/\ell > 1/\sqrt{2}$, and negative if $r_+/\ell < 1/\sqrt{2}$. This means that small AdS black holes are unstable and can evaporate, whereas large AdS black holes are stable and do not evaporate.

5. Plotting F as a function of β (for various values of r_+) can be found in Fig. 1. The small black holes correspond to the part of the curve connected to the origin up to the phase transition, the large black holes are the ones that go to negative F. Empty AdS has $F = 0$ for all values of β. At very low temperatures (large β), the thermodynamically preferred configuration (lowest F) is thus empty AdS. If we increase T (lower β), at one point the large black holes will start to dominate. This is the *Hawking-Page phase transition.*

2 Exercise session 2

2.1 Von Neumann entropy and Rényi entropies

1. We have
$$\begin{aligned}\rho_{12} &= \frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\\ \rho_1 &= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned} \tag{53}$$

These are the same, so the rest of the computation will be identical for both of them. We have twice the eigenvalue $\lambda = 1/2$ so

$$S(\rho_{1/12}) = \log 2. \tag{54}$$

We have

$$\rho_{1/12}^n = \frac{1}{2^n}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \operatorname{Tr} \rho_{1/12}^n = 2^{1-n}. \tag{55}$$

So then

$$S^{(n)}(\rho_{1/12}) = \log 2. \tag{56}$$

2. We have

$$\begin{aligned} \rho_{12} &= \frac{|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 10|}{3}, \\ \rho_1 &= \frac{|1\rangle\langle 1| + 2|0\rangle\langle 0|}{3}. \end{aligned} \tag{57}$$

We can write this as matrices as

$$\rho_{12} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \rho_1 = \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}. \tag{58}$$

We can compute the eigenvalues of these as

$$\det(\rho_{12}\text{-}\lambda\mathbb{1}) = \left(\frac{1}{3} - \lambda\right)\left(\left(\frac{1}{3} - \lambda\right)^2 - \frac{1}{9}\right) = -\lambda\left(\lambda - \frac{2}{3}\right)\left(\lambda - \frac{1}{3}\right). \tag{59}$$

This means that we have

$$S(\rho_{12}) = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = \log 3 - \frac{2}{3}\log 2. \tag{60}$$

For ρ_1, the eigenvalues are $\lambda = \frac{1}{3}, \frac{2}{3}$ such that

$$S(\rho_1) = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = \log 3 - \frac{2}{3}\log 2. \tag{61}$$

For the Rényi entropies we use

$$\rho_{12}^n = \left(\frac{1}{3}\right)^n \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{n-1} & 2^{n-1} \\ 0 & 2^{n-1} & 2^{n-1} \end{pmatrix} \to \operatorname{Tr} \rho_{12}^n = \left(\frac{1}{3}\right)^n (2^n + 1). \tag{62}$$

so

$$S^{(n)}(\rho_{12}) = \frac{n}{n-1}\log 3 - \frac{1}{n-1}\log(2^n + 1). \tag{63}$$

Also

$$\rho_1^n = \frac{1}{3^n}\begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} \to \operatorname{Tr} \rho_1^n = \frac{2^n + 1}{3^n}, \tag{64}$$

so also for ρ_1 we have

$$S^{(n)}(\rho_1) = \frac{n}{n-1}\log 3 - \frac{1}{n-1}\log(2^n + 1). \tag{65}$$

3. For the GHZ state, the $n \to 1$ limit is trivial:

$$\lim_{n\to 1} \log 2 = \log 2. \tag{66}$$

For the W state, it is a bit more non-trivial. We have

$$\lim_{n\to 1} \frac{n \log 3 - \log(2^n + 1)}{n-1} = \lim_{n\to 1} \frac{\log 3 - \frac{2^n \log 2}{2^n+1}}{1} = \log 3 - \frac{2}{3}\log 2. \tag{67}$$

Here we used l'Hopitals rule, since both the numerator and the denominator approach zero. Interestingly, both limits equal the respective von Neumann entropies!

4. We have

$$\operatorname{Tr} \rho^n = \sum_i \lambda_i^n = \sum_i \lambda_i \lambda_i^{n-1}. \tag{68}$$

We can use $\lambda_1 = \sum_i \lambda_i$ and add and subtract 1 to find

$$\operatorname{Tr} \rho^n = 1 + \sum_i \lambda_i(\lambda_i^{n-1} - 1). \tag{69}$$

5. As $n \to 1$, the term inside the sum goes to zero. So we can take the limit

$$\lim_{n\to 1} \log \operatorname{Tr} \rho^n = \lim_{n\to 1} \sum_i \lambda_i(\lambda_i^{n-1} - 1). \tag{70}$$

So we see

$$\lim_{n\to 1} S^{(n)} = \lim_{n\to 1} \frac{\sum_i \lambda_i(\lambda_i^{n-1} - 1)}{1-n} = \lim_{n\to 1} -\sum_i \lambda_i \frac{\lambda_i^{n-1}\log\lambda_i}{\lambda_i^{n-1}} = -\sum_i \lambda_i \log \lambda_i = S(\rho). \tag{71}$$

6. We find

$$-\partial_n \operatorname{Tr} \rho^n|_{n=1} = -\sum_i \lambda_i^n \log \lambda_i|_{n=1} = -\sum_i \lambda_i \log\lambda_i = S(\rho). \tag{72}$$

2.2 CFT two-point function

1. We write

$$\langle \mathcal{O}(\tau_1, x_1)\mathcal{O}(\tau_2, x_2)\rangle = \langle 0| e^{\tau_1 H}\mathcal{O}(x_1)e^{(\tau_2-\tau_1)H}\mathcal{O}(x_2)e^{-\tau_2 H}|0\rangle. \tag{73}$$

Since the Hamiltonian acting on the vacuum is zero, this is

$$\langle 0| \mathcal{O}(x_1)e^{(\tau_2-\tau_1)H}\mathcal{O}(x_2)|0\rangle. \tag{74}$$

We assume that $\mathcal{O}$ is not very special in that it does not create only specific energy eigenvalues. That means that

$$\mathcal{O}(x_2)|0\rangle = \int_0^\infty dE f(E, x_2)|E\rangle. \tag{75}$$

This means that we have

$$\int_0^\infty dE f(E, x_2) e^{(\tau_2-\tau_1)E}\langle 0|\mathcal{O}(x_1)|E\rangle. \tag{76}$$

This means that for $\tau_2 - \tau_1 > 0$, unless $f(E)$ decreases exponentially (which it generically does not do), we have an exponential divergence. If $\tau_1 > \tau_2$, it has an exponential damping, so it is bounded. This tells us that in Euclidean space, we should only consider time-ordered correlation functions.

2. Taking $\tau = it$,

$$\langle \mathcal{O}(it, x)\mathcal{O}(0)\rangle = \frac{1}{(x^2 - t^2)^\Delta}. \tag{77}$$

There are singularities at $t = \pm|x|$.

3. Writing $\tau = it + \epsilon$, we have

$$x^2 + \tau^2 = x^2 - t^2 + 2it\epsilon. \tag{78}$$

Writing this as a complex number as $re^{i\phi}$, we can compute

$$\log(x^2 - t^2 + 2it\epsilon) = \log r + i\phi = \log|x^2 - t^2| + i\arg(x^2 - t^2 + 2it\epsilon). \tag{79}$$

The logarithm has a branch cut. We place it at the negative real axis. Then if $x^2 - t^2 > 0$ (spacelike separation), we have arg $= 0$. If $x^2 - t^2 < 0$ (timelike separation), it depends on the sign of t (we take $\epsilon \to 0^+$). If $t > 0$, the argument is π. If $t < 0$, the argument is $-\pi$. So

$$\langle \mathcal{O}(it, x)\mathcal{O}(0)\rangle = \frac{1}{|x^2 - t^2|^\Delta}\begin{cases} e^{-i\pi\Delta} & \text{timelike, } t > 0, \\ 1 & \text{spacelike,} \\ e^{i\pi\Delta} & \text{timelike}, t < 0. \end{cases} \tag{80}$$

Another interesting thing to note is that in Lorentzian correlators, we can consider both time-ordered and anti-time ordered correlators. Which one corresponds to the Euclidean one? We see that if $\epsilon > 0$, writing $\tau = it - \epsilon$ so $t = -i\tau + \epsilon$,

$$\langle \mathcal{O}_1(t, x)\mathcal{O}_2(0)\rangle_L = \langle \mathcal{O}_1(-i\tau + \epsilon, x)\mathcal{O}_2(0)\rangle_E. \tag{81}$$

This is correctly time-ordered, since $\epsilon > 0$ so the Euclidean operators are in the right order. For anti-time-ordered, taking $\tau = it + \epsilon$ with again $\epsilon > 0$,

$$\langle \mathcal{O}_1(0)\mathcal{O}_2(t, x)\rangle_L = \langle \mathcal{O}_1(0)\mathcal{O}_2(-it - \epsilon)\rangle_E. \tag{82}$$

So we see that time-ordered Lorentzian correlators correspond to the $-\epsilon$ prescription ($\tau = it - \epsilon$), whereas anti-time-ordered Lorentzian correlators correspond to the $+\epsilon$ prescription ($\tau = it + \epsilon$).

2.3 Wave propagation in Schwarzschild spacetime

The metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{83}$$

The wave equation is

$$\frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}g^{ij}\partial_j\Phi\right) = 0 \tag{84}$$

1. Explicitly written out in coordinates, using that $\sqrt{-g} = r^2\sin\theta$, we get the wave equation

$$\begin{aligned} &\frac{1}{r^2\sin\theta}\partial_r\left(r^2\sin\theta f(r)\partial_r\Phi\right) + \frac{1}{r^2\sin\theta}\partial_t\left(\frac{r^2\sin\theta}{-f(r)}\partial_t\Phi\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(r^2\sin\theta\frac{1}{r^2}\partial_\theta\Phi\right) \\ &+ \frac{1}{r^2\sin\theta}\partial_\varphi\left(r^2\sin\theta\frac{1}{r^2\sin^2\theta}\partial_\varphi\Phi\right) = 0, \\ &\frac{1}{r^2}\partial_r\left(r^2 f(r)\partial_r\Phi\right) - \frac{1}{f(r)}\partial_t^2\Phi + \frac{1}{r^2}\partial_\theta^2\Phi + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\Phi = 0. \end{aligned} \tag{85}$$

2. Writing
$$\Phi = e^{-i\omega t} Y_{lm}(\theta,\varphi)\phi_{\omega lm}(r), \tag{86}$$
we note
$$\partial_t^2\Phi = -\omega^2\Phi, \quad \frac{1}{r^2}\partial_\theta^2\Phi + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\Phi = -\frac{l(l+1)}{r^2}\Phi. \tag{87}$$
Then the wave equation reduces to
$$\frac{1}{r^2}\partial_r(r^2 f(r)\partial_r\phi) + \frac{\omega^2}{f}\phi - \frac{l(l+1)}{r^2}\phi = 0. \tag{88}$$

3. Now we write
$$\phi = \frac{\psi}{r}, \quad dr_* = \frac{dr}{f(r)}. \tag{89}$$
Then
$$\partial_r\frac{\psi}{r} = -\frac{\psi}{r^2} + \frac{1}{r}\frac{dr_*}{dr}\psi' = -\frac{\psi}{r^2} + \frac{\psi'}{rf}. \tag{90}$$
Here we write $\psi = \psi(r_*)$, so ψ' is with the derivative with respect to r_*. However, we will keep $f = f(r)$, so f' means the derivative with respect to r. Then
$$\partial_r\left(r^2 f\partial_r\frac{\psi}{r}\right) = \partial_r\left(-\psi f + r\psi'\right) = -\psi f' - \psi' + \psi' + \frac{r}{f}\psi'' = -\psi f' + \frac{r}{f}\psi''. \tag{91}$$
The wave equation becomes
$$-\frac{\psi f'}{r^2} + \frac{\psi''}{fr} + \frac{\omega^2}{fr}\psi - \frac{l(l+1)}{r^3}\psi = 0. \tag{92}$$
This can be rewritten to
$$-\psi'' = \left(\omega^2 - f\left(\frac{f'}{r} + \frac{l(l+1)}{r^2}\right)\right)\psi. \tag{93}$$
So we have
$$V_l(r) = f\left(\frac{f'}{r} + \frac{l(l+1)}{r^2}\right). \tag{94}$$

4. For specified f, one can plot V as a function of r (or r_*). For the Schwarzschild metric, $f(r) = 1 - \frac{2GM}{r}$, this is visualised in Fig. 2.

5. It can be seen that this potential does not have any minima. Because of that, there cannot be any bound states. Furthermore, the effective potential has a maximum. For ω much smaller than this potential maximum, the waves get mostly reflected back to infinity, while for ω much larger, the waves get absorbed by the black hole.

3 Exercise session 3

3.1 Holographic 4-pt functions: how to succeed without really trying

1. Just match the subscripts of the functions to the lines!

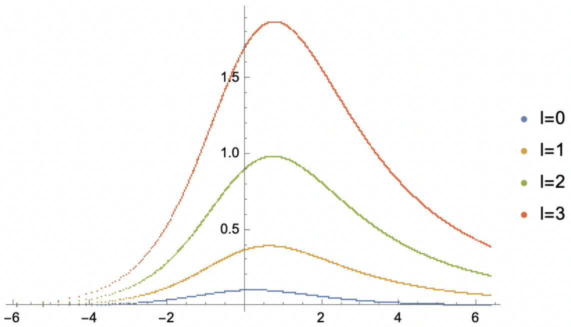

Figure 2: Effective potential V_l as a function of r_*.

2. We can write
$$z_0^2 + |\mathbf{z} - \mathbf{z}'|^2 = (z - z')^2. \tag{95}$$
Under inversion, this becomes
$$(\bar{z} - \bar{z}')^2 = \frac{z^2}{z^4} + \frac{(z')^2}{(z')^4} - \frac{2zz'}{z^2(z')^2} = \frac{(z-z')^2}{z^2(z')^2}. \tag{96}$$
This means that
$$\mathcal{G}(\bar{z}_0, \bar{\mathbf{z}}, \bar{\mathbf{z}'}) = \left(\frac{z_0^2}{z^2}\frac{z^2(z')^2}{(z-z')^2}\right)^{\Delta} = \mathcal{G}(z_0, \mathbf{z}, \mathbf{z}')|\mathbf{z}'|^{2\Delta}. \tag{97}$$

3. We first use boundary translation invariance to send $z_1 \to 0$ and $z_3 \to z_{31}$.
$$\begin{aligned} A_3(w, \mathbf{z}_1, \mathbf{z}_3) &= \int \frac{d^{d+1}u}{u_0^{d+1}} G_\Delta(\xi)\mathcal{G}_{\Delta_1}(u,0)\mathcal{G}_{\Delta_3}(u, z_{31}) \\ &= \int \frac{d^{d+1}u}{u_0^{d+1}} G_\Delta(\xi)\left(\frac{u_0}{u^2}\right)^{\Delta_1}\mathcal{G}_{\Delta_3}(u, z_{31} \end{aligned} \tag{98}$$
Note that ξ is invariant under inversion:
$$\bar{\xi} = \frac{(\bar{z} - \bar{w})^2}{2\bar{w}_0\bar{z}_0} = \frac{(z-w)^2}{2w_0z_0} = \xi. \tag{99}$$
So we see that
$$A_3(w, \mathbf{z}_1, \mathbf{z}_3) = |\mathbf{z}_{13}|^{-2\Delta_3}\int \frac{d^{d+1}u}{u_0^{d+1}} G_\Delta(\xi)\,(u_0)^{\Delta_1}\left(\frac{u_0}{(u - \bar{z}_{31})^2}\right)^{\Delta_3}. \tag{100}$$
Now we take $u \to u + \bar{z}_{31} = u - \bar{z}_{13}$. This keeps the integral measure invariant, but ξ changes. However, if we send $w \to w - \bar{z}_{13}$ as well, then ξ is invariant again. So we see that
$$A_3(w, \mathbf{z}_1, \mathbf{z}_3) = |\mathbf{z}_{13}|^{-2\Delta_3} I(\bar{w} - \bar{\mathbf{z}}_{13}) \tag{101}$$
where
$$I(w) = \int \frac{d^{d+1}u}{u_0^{d+1}} G_\Delta(\xi)\,(u_0)^{\Delta_1}\left(\frac{u_0}{u^2}\right)^{\Delta_3} \tag{102}$$

4. Poincaré transformations on the boundary do not transform w_0 and u_0. It also keeps $(u-w)^2$ invariant. So this means that ξ is invariant under these transformations. u_0 is again not transformed under these and u^2 is also manifestly Poincaré invariant.

5. Scaling both u and w gives $(u-w)^2 \to \lambda^2(u-w)^2$, this cancels exactly against $w_0 z_0 \to \lambda^2 w_0 z_0$. So the chord distance is invariant if we also rescale u. The integral measure is also invariant under this. u_0^Δ transforms to $\lambda^{\Delta_1} u_0^{\Delta_1}$ and $\left(\frac{u_0}{u^2}\right)^{\Delta_3}$ goes to $\lambda^{-\Delta_3}\left(\frac{u_0}{u^2}\right)^{\Delta_3}$ so we see that

$$I(\lambda w) = \lambda^{\Delta_{13}} I(w). \tag{103}$$

6. The function that is invariant under both rescaling and Poincaré transformations is $s \equiv \frac{w_0^2}{w^2}$. So any function of this will be invariant under the transformations mentioned above. We still want total invariance under the Poincaré subgroup, but a specific scaling law, so the function will in general be of the form

$$I(w) = w_0^{\Delta_{13}} f(s). \tag{104}$$

7. We want to show that

$$\left(-\nabla^2 + m^2\right) I(w) = w_0^{\Delta_{13}} \left(\frac{w_0^2}{w^2}\right)^{\Delta_3}. \tag{105}$$

We can use the fact that

$$(-\nabla^2 + m^2) G_\Delta(\xi) = \frac{\delta(u-w)}{\sqrt{g}} = \delta(u-w) u_0^{d+1}. \tag{106}$$

This means that

$$\begin{aligned}\left(-\nabla^2 + m^2\right) I(w) &= \int \frac{d^{d+1}u}{u_0^{d+1}} (-\nabla^2 + m^2) G_\Delta(\xi) u_0^{\Delta_1} \left(\frac{u_0}{u^2}\right)^{\Delta_3} \\ &= \int \frac{d^{d+1}u}{u_0^{d+1}} \delta(u-w) u_0^{d+1} u_0^{\Delta_1} \left(\frac{u_0}{u^2}\right)^{\Delta_3} \\ &= w_0^{\Delta_1} \left(\frac{w_0}{w^2}\right)^{\Delta_3} = w_0^{\Delta_{13}} \left(\frac{w_0^2}{w^2}\right)^{\Delta_3}.\end{aligned} \tag{107}$$

8. You should find the correct result after working precisely or plugging this into Mathematica. It's taking derivatives.

9. $s \to 1$ corresponds to $\mathbf{w} \to 0$. This is not at all a special point in the bulk, so f should be smooth there.

10. As $w_0 \to 0$, $\xi \sim w_0^{-\Delta}$. The hypergeometric function approaches 1 as $1/\xi^2$ approaches zero, so $G \sim w_0^\Delta$. Since we need $w_0^{\Delta_{13}} f(s) \sim w_0^\Delta$, we need $f(s \to 0) \sim w_0^{\Delta - \Delta_{13}}$ so

$$f(s \to 0) \sim s^{\frac{\Delta - \Delta_{13}}{2}}. \tag{108}$$

11. The differential equation reduces to

$$4s^2(s-1) f'' + 4s\left[(\Delta_{13}+1)s - \Delta_{13} + \frac{D}{2} - 1\right] f' + \left[\Delta_{13}(D-\Delta_{13}) + m^2\right] f = s^{\Delta_3}. \tag{109}$$

Plugging in $f = \sum_l a_l s^{\Delta_3+l}$ gives

$$\sum_l a_l \Bigg(4(s-1)(\Delta_3+l)(\Delta_3+l-1)s^{\Delta_3+l} + 4\left[(\Delta_{13}+1)s - \Delta_{13} + \frac{D}{2} - 1\right](\Delta_3+l)s^{\Delta_3+l} + \left[\Delta_{13}(D-\Delta_{13}) + m^2\right] s^{\Delta_3+l} \Bigg) = s^{\Delta_3} \tag{110}$$

This gives

$$\sum_l a_l \Bigg(4(s-1)(\Delta_3+l)(\Delta_3+l-1)s^{l} + 4\left[(\Delta_{13}+1)s - \Delta_{13} + \frac{D}{2} - 1\right](\Delta_3+l)s^{l} + \left[\Delta_{13}(D-\Delta_{13}) + m^2\right] s^{l} \Bigg) = 1 \tag{111}$$

We will consider only fixed powers of s. For order n in l, we will have contributions from both a_n and a_{n-1}. To zeroth order in s, we should have exactly 1. This will get contributions from $l = 0$ and $l = -1$. So:

$$a_{-1}\left(4(\Delta_3-1)(\Delta_3-2) + 4(\Delta_{13}+1)(\Delta_3-1)\right) = 4a_{-1}\left(-\Delta_3 + 1 + \Delta_1\Delta_3 - \Delta_1\right) = 1 \tag{112}$$

This means that

$$a_{-1} = \frac{1}{4(1-\Delta_1)(1-\Delta_3)} \tag{113}$$

For lower orders of s we see

$$\begin{aligned} &a_{k-1}\left(4(\Delta_3+k-1)(\Delta_3+k-2) + 4(\Delta_{13}+1)(\Delta_3+k-1)\right) \\ &= -a_k\left(-4(\Delta_3+k)(\Delta_3+k-1) - 4(\Delta_3+k)(-\Delta_{13} + \frac{D}{2} - 1) + \Delta_{13}(D-\Delta_{13}+m^2)\right) \end{aligned} \tag{114}$$

Using the fact that $\Delta = \frac{D}{2} + \frac{1}{2}\sqrt{D^2+4m^2}$, you should be able to rewrite this to the formula given.

12. If one of the a_k ever becomes zero, then everything below that will also be zero. This means that either

$$l = -\frac{\Delta_1 + \Delta_3 - \Delta}{2} \tag{115}$$

or

$$l = -\frac{\Delta_1 + \Delta_3 + \Delta - D}{2}. \tag{116}$$

So it terminates if either $\Delta_1 + \Delta_3 - \Delta$ is a positive, even number or if $\Delta_1 + \Delta_3 + \Delta - D$ is a positive, even number. However, we want

$$l + \Delta_3 = \frac{\Delta - \Delta_{13}}{2}, \tag{117}$$

so we pick the first option.

13. We just plug in the result:

$$A_3 = |\mathbf{z}_{13}|^{-2\Delta_3} I(\bar{w} - \bar{\mathbf{z}}_{13}) = |\mathbf{z}_{13}|^{-2\Delta_3} \bar{w_0}^{\Delta_{13}} \sum_k a_k \left(\frac{\bar{w_0}^2}{(\bar{w} - \bar{\mathbf{z}}_{13})^2} \right)^{\Delta_3+k}$$
$$= |\mathbf{z}_{13}|^{-2\Delta_3} \left(\frac{w_0}{w^2} \right)^{\Delta_{13}} \sum_k a_k \left(\frac{w_0^2 \mathbf{z}_{13}^2}{w^2 (w - \mathbf{z}_{13})^2} \right)^{\Delta_3+k}. \tag{118}$$

and we get

$$A_4 = \sum_l a_l \int \frac{d^{D+1}w}{w_0^{D+1}} \mathcal{G}_{\Delta_2}(w, \mathbf{z}_2) \mathcal{G}_{\Delta_4}(w, \mathbf{z}_4) |\mathbf{z}_{13}|^{-2\Delta_3} \left(\frac{w_0}{w^2} \right)^{\Delta_{13}} \left(\frac{w_0^2 z_{13}^2}{w^2 (w - z_{13})^2} \right)^{\Delta_3+l}$$
$$= \sum_l a_l \int \frac{d^{D+1}w}{w_0^{D+1}} \mathcal{G}_{\Delta_2}(w, \mathbf{z}_2) \mathcal{G}_{\Delta_4}(w, \mathbf{z}_4) |\mathbf{z}_{13}|^{2l} \left(\frac{w_0}{w^2} \right)^{\Delta_{13}} \left(\frac{w_0^2}{w^2 (w - z_{13})^2} \right)^{\Delta_3+l} \tag{119}$$

Now

$$\mathcal{G}_{\Delta_1+l}(w, 0) \mathcal{G}_{\Delta_3+l}(w, \mathbf{z}_{13}) = \left(\frac{w_0}{w^2} \right)^{\Delta_1+l} \left(\frac{w_0}{(w - z_{13})^2} \right)^{\Delta_3+l}$$
$$= w_0^{\Delta_1-\Delta_3} \left(\frac{1}{w^2} \right)^{\Delta_1+l} \left(\frac{w_0^2}{(w - z_{13})^2} \right)^{\Delta_3+l} \tag{120}$$
$$= \left(\frac{w_0}{w^2} \right)^{\Delta_{13}} \left(\frac{w_0^2}{w^2 (w - z_{13})^2} \right)^{\Delta_3+l}.$$

So we see that we indeed have

$$A_4 = \sum_l a_l |\mathbf{z}_{13}|^{2l} \int \frac{d^{D+1}w}{w_0^{D+1}} \mathcal{G}_{\Delta_1+l}(w, 0) \mathcal{G}_{\Delta_3+l}(w, \mathbf{z}_{13}) \mathcal{G}_{\Delta_2}(w, \mathbf{z}_2) \mathcal{G}_{\Delta_4}(w, \mathbf{z}_4). \tag{121}$$

3.2 3d gravity as group theory

1. We have

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{122}$$

This means that

$$e^{\frac{i}{2}(t+\phi)\sigma_2} = \sum_{n=0}^{\infty} \frac{(i(t+\phi))^n}{n!} \sigma_2^n = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{t+\phi}{2}\right)^{2n}}{2n!} \mathbb{1} + i \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{t+\phi}{2}\right)^{2n+1}}{(2n+1)!} \sigma_2$$
$$= \cos\left(\frac{t+\phi}{2}\right) \mathbb{1} + i \sin\left(\frac{t+\phi}{2}\right) \sigma_2 = \begin{pmatrix} \cos\left(\frac{t+\phi}{2}\right) & \sin\left(\frac{t+\phi}{2}\right) \\ -\sin\left(\frac{t+\phi}{2}\right) & \cos\left(\frac{t+\phi}{2}\right) \end{pmatrix}. \tag{123}$$

We also have

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{124}$$

Then

$$
\begin{aligned}
e^{\rho\sigma_3} &= \sum_{n=0}^{\infty} \frac{\rho^n}{n!}\sigma_3^n = \sum_{n=0}^{\infty} \frac{\rho^{2n}}{2n!}\mathbb{1} + \sum_{n=0}^{\infty} \frac{\rho^{2n+1}}{(2n+1)}\sigma_3 \\
&= \cosh\rho\mathbb{1} + \sinh\rho\sigma_3 = \begin{pmatrix} \cosh\rho + \sinh\rho & 0 \\ 0 & \cosh\rho - \sinh\rho \end{pmatrix}.
\end{aligned} \tag{125}
$$

Then

$$
\begin{aligned}
&e^{\frac{i}{2}(t+\phi)\sigma_2} e^{\rho\sigma_3} e^{\frac{i}{2}(t-\phi)\sigma_2} \\
&= \begin{pmatrix} \cos\left(\frac{t+\phi}{2}\right) & \sin\left(\frac{t+\phi}{2}\right) \\ -\sin\left(\frac{t+\phi}{2}\right) & \cos\left(\frac{t+\phi}{2}\right) \end{pmatrix} \begin{pmatrix} \cosh\rho + \sinh\rho & 0 \\ 0 & \cosh\rho - \sinh\rho. \end{pmatrix} \begin{pmatrix} \cos\left(\frac{t-\phi}{2}\right) & \sin\left(\frac{t-\phi}{2}\right) \\ -\sin\left(\frac{t-\phi}{2}\right) & \cos\left(\frac{t-\phi}{2}\right) \end{pmatrix} \\
&= \begin{pmatrix} \cos t\cosh\rho + \cos\phi\sinh\rho & \sin t\cosh\rho - \sin\phi\sinh\rho \\ -\sin t\cosh\rho - \sin\phi\sinh\rho & \cos t\cosh\rho - \cos\phi\sinh\rho \end{pmatrix}.
\end{aligned} \tag{126}
$$

2. The group parametrized by 2x2 real matrices with determinant 1 is $SL(2,\mathbb{R})$.
3. Plugging it into Mathematica will show you!
4. We compute

$$
\begin{aligned}
e^{\varphi\sigma_3} &= \sum_{n=0}^{\infty} \frac{\varphi^n}{n!}\sigma_3^n = \sum_{n=0}^{\infty} \frac{\varphi^n}{n!} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix} \\
&= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{\varphi^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{(-\varphi)^n}{n!} \end{pmatrix} = \begin{pmatrix} e^{\varphi} & 0 \\ 0 & e^{-\varphi} \end{pmatrix}.
\end{aligned} \tag{127}
$$

The same goes for $e^{\psi\sigma_3}$. Lastly,

$$
\begin{aligned}
e^{\rho\sigma_1} &= \sum_{n=0}^{\infty} \frac{\rho^{2n}}{(2n)!}\mathbb{1} + \sum_{n=0}^{\infty} \frac{\rho^{2n+1}}{(2n+1)!}\sigma_1 \\
&= \begin{pmatrix} \cosh\rho & \sinh\rho \\ \sinh\rho & \cosh\rho \end{pmatrix} = \begin{pmatrix} r & \sqrt{r^2-1} \\ \sqrt{r^2-1} & r \end{pmatrix}.
\end{aligned} \tag{128}
$$

Here we used $r = \cosh\rho$, $\sinh\rho = \sqrt{\cosh^2\rho - 1}$ for ρ, r positive. It is easy to check that all three matrices have determinant 1, so the product of all three will also have determinant 1.

5. Again, simply plugging into Mathematica gives the desired result.

4 Exercise session 4

4.1 Black hole information paradox

1. For the four-dimensional Reissner-Nordström black hole, the metric is given by

$$
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \qquad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{129}
$$

The horizons are at

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{130}$$

We will work in the near-horizon, near-extremal limit. That means that we will take $M = Q + \Delta M$ with ΔM small. This gives $r_\pm \sim |Q|$ (we will take Q positive from now on). The near-horizon limit we will take as $r = Q + Q^2\tilde{r}$, with $Q^2\tilde{r}$ small. Then

$$\begin{aligned} f(r) &= 1 - \frac{2Q + 2\Delta M}{r} + \frac{Q^2}{r^2} \\ &= \frac{(r-Q)^2}{r^2} - \frac{2\Delta M}{r} \approx Q^2\tilde{r}^2 - \frac{2\Delta M}{Q}. \end{aligned} \tag{131}$$

We also have $dr^2 = Q^4 d\tilde{r}^2$, so

$$ds^2 \to Q^2\left(-\left(\tilde{r}^2 - \frac{2M}{Q^3}\right)dt^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 - \frac{2M}{Q^3}} + d\Omega_2^2\right). \tag{132}$$

This tells us that the near-horizon, near extremal limit of the Reissner-Nordström black hole looks like a black hole on $\mathrm{AdS}_2 \times S^2$.

2. Writing $t = f(u)$ and $z = z(u)$, we find

$$g_{uu} = \frac{-(f')^2 + (z')^2}{z^2} = \frac{-1}{\epsilon^2}. \tag{133}$$

Since the boundary particle is near the boundary, we can assume z is small. We will also assume that its trajectory is not too wildly fluctuating, such that z' is also small. Then

$$g_{uu} \approx -\frac{(f')^2}{z^2} = -\frac{1}{\epsilon^2}. \tag{134}$$

We see that for any f, we can solve this by setting $z = \epsilon f'$. Then we go to the dilaton profile. Since $x^+x^- = t^2 - z^2 \approx f^2$ and $x^+ - x^- = 2z = 2\epsilon f'$, we find

$$\phi|_{\text{bdy}} \approx \bar{\phi}_r \frac{1 - (\pi T_0)^2 f^2}{\epsilon f'} = \frac{\bar{\phi}_r}{\epsilon}. \tag{135}$$

To solve this, we need $f' = 1 - (\pi T_0)^2 f^2$. We can solve this through

$$\begin{aligned} \int \frac{df}{1 - (\pi T_0)^2 f^2} &= \int du, \\ \frac{1}{\pi T_0}\int \frac{d\tilde{f}}{1 - \tilde{f}^2} &= \frac{\operatorname{arctanh}\tilde{f}}{\pi T_0} = u, \\ \tilde{f} &= \tanh(\pi T_0 u), \\ f(u) &= \frac{1}{\pi T_0}\tanh(\pi T_0 u). \end{aligned} \tag{136}$$

3. The boundary is parameterized through $t = f(u)$, $z = \epsilon f'(u)$. This means that we can write

$$\begin{aligned} f(\bar{y}) &= \bar{x} = t + z = f(u) + \epsilon f'(u) \approx f(u+\epsilon), \\ f(y) &= x = -t + z = -f(u) + \epsilon f'(u) \approx -f(u-\epsilon). \end{aligned} \tag{137}$$

Assuming $f(-y) = -f(y)$, we derive from here that the boundary in $y, \bar{y}$ coordinates is given by

$$\bar{y} = u + \epsilon, y = -(u - \epsilon). \tag{138}$$

That means that $y + \bar{y} = 2\epsilon$ is the boundary location, and the time there is given by $\frac{\bar{y}-y}{2} = u$.

4. Möbius transformations are defined as

$$w(x) = \frac{ax+b}{cx+d}, \qquad ad - bc \neq 0. \tag{139}$$

Then

$$\begin{aligned} w' &= \frac{(cx+d)a - (ax+b)c}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}, \\ w'' &= -2(ad-bc)\frac{c}{(cx+d)^3} \\ w''' &= 6c^2(ad-bc)\frac{1}{(cx+d)^4}. \end{aligned} \tag{140}$$

Therefore we see that

$$\{w, x\} = \frac{6c^2}{(cx+d)^2} - \frac{3}{2}\frac{4c^2}{(cx+d)^2} = 0. \tag{141}$$

5. A BCFT two-point function is computed by a CFT four-point function, with mirrored operators and half the conformal dimensions. The mirrored operator from $(x, \bar{x})$ has coordinates $(-\bar{x}, -x)$

$$\langle\sigma(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\rangle_{\text{BCFT},\Delta} = \langle\sigma(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\sigma(-\bar{x}_1,-x_1)\sigma(-\bar{x}_2,-x_2)\rangle_{\text{CFT},\Delta/2}.$$

A conformal primary of dimension Δ scales under a conformal transformation as

$$\mathcal{O}(x)_{\Omega^{-2}g} = \Omega(x)^{\Delta}\mathcal{O}(x)_g. \tag{142}$$

This means that

$$\begin{aligned} &\langle\sigma(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\rangle_{\text{BCFT},\Delta,\Omega^{-2}g} \\ &= \Omega(x_1,\bar{x}_1)^{\Delta/2}\Omega(x_2,\bar{x}_2)^{\Delta/2}\Omega(-\bar{x}_1,-x_1)^{\Delta/2}\Omega(-\bar{x}_2,-x_2)^{\Delta/2}\langle\sigma(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\rangle_{\text{BCFT},\Delta,g} \\ &= \Omega(x_1,\bar{x}_1)^{\Delta}\Omega(x_2,\bar{x}_2)^{\Delta}\langle\sigma(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\rangle_{\text{BCFT},\Delta,g}. \end{aligned}$$

This means that

$$S^{(n)}_{\Omega^{-2}g} = -\frac{1}{n-1}\log\left(\Omega(x_1,\bar{x}_1)^{\Delta}\Omega(x_2,\bar{x}_2)^{\Delta}\right) + S_g = -\frac{c}{12}\frac{n+1}{n}\sum_{\text{endpoints}}\log\Omega + S^{(n)}_g. \tag{143}$$

This gives us

$$S_{\Omega^{-2}g} = S_g - \frac{c}{6}\sum_{\text{endpoints}}\log\Omega. \tag{144}$$

6. The conformal cross-ratio is given by

$$\eta = \frac{(w_1+\bar{w}_1)(w_2+\bar{w}_2)}{(w_1+\bar{w}_2)(w_2+\bar{w}_1)}. \tag{145}$$

We write $w = Z - T$ and $\bar{w} = Z + T$. A translation in T very obviously cancels out in the conformal cross ratio. The BCFT is not invariant under transformations in Z (orthogonal to the boundary), so we don't have to check that one. Neither is it invariant under rotations. We are left with scale transformations and inversions. A scaling also obviously cancels out. An inversion gives

$$\eta \to \frac{(\frac{1}{\bar{w}_1} + \frac{1}{w_1})(\frac{1}{\bar{w}_2} + \frac{1}{w_2})}{(\frac{1}{\bar{w}_1} + \frac{1}{w_2})(\frac{1}{\bar{w}_2} + \frac{1}{w_1})} = \frac{(w_1 + \bar{w}_1)(w_2 + \bar{w}_2)}{(w_2 + \bar{w}_1)(w_1 + \bar{w}_2)} = \eta. \tag{146}$$

7. Both endpoints to the past of the shock means $x^+_{1,2} > 0$ and $x^-_{1,2} < 0$. This means that $x_{1,2}, \bar{x}_{1,2} > 0$. We take one of the endpoints all the way into the AdS. In the w-space, this means that one of the endpoints is at $w = 0$, the boundary. So we need to use the formula

$$S_g = \frac{c}{6} \log(w + \bar{w}) + \log g. \tag{147}$$

We need to transform to our metric, so we actually need the formula

$$S = \frac{c}{6} \log(w + \bar{w}) + \log g - \frac{c}{6} \log \Omega(w, \bar{w}). \tag{148}$$

Since

$$\frac{4dxd\bar{x}}{(x + \bar{x})^2} = \frac{4w'(x)\bar{w}'(\bar{x})}{(x + \bar{x})^2} dwd\bar{w} \quad \Longrightarrow \quad \Omega(w, \bar{w}) = \frac{x + \bar{x}}{2} \sqrt{w'(x)\bar{w}'(\bar{x})}. \tag{149}$$

Since $x, \bar{x} > 0$ we use

$$\begin{aligned} w &= \frac{w_0^2}{x} \quad &\to \quad w' &= -\frac{w_0^2}{x^2}, \\ \bar{w} &= \frac{w_0^2}{\bar{x}} \quad &\to \quad \bar{w}' &= -\frac{w_0^2}{\bar{x}^2}, \end{aligned} \tag{150}$$

so

$$\Omega = \frac{x + \bar{x}}{2} \sqrt{\frac{w_0^4}{x^2 \bar{x}^2}} = \frac{w_0^2}{2} \left(\frac{1}{\bar{x}} + \frac{1}{x} \right) = \frac{w + \bar{w}}{2}. \tag{151}$$

Therefore,

$$S = \frac{c}{6} \log 2 + \log g. \tag{152}$$

8. We take one endpoint to the future (x_1) and one to the past (x_2). This means $x_1 < 0$, $\bar{x}_1 > 0$, $x_2, \bar{x}_2 > 0$ We first compute the conformal cross-ratio. We have (we transform to lightcone coordinates already)

$$w_1 = f^{-1}(x_1^-), \quad \bar{w}_1 = \frac{w_0^2}{x_1^+}, \quad w_2 = -\frac{w_0^2}{x_2^-}, \quad \bar{w}_2 = \frac{w_0^2}{x_2^+}. \tag{153}$$

Then the conformal cross-ratio is

$$\eta = \frac{\left(f^{-1}(x_1^-) + \frac{w_0^2}{x_1^+} \right) \left(\frac{w_0^2}{x_2^+} - \frac{w_0^2}{x_2^-} \right)}{\left(f^{-1}(x_1^-) + \frac{w_0^2}{x_2^+} \right) \left(\frac{w_0^2}{x_1^+} - \frac{w_0^2}{x_2^-} \right)} \approx \frac{\left(\frac{1}{x_2^+} - \frac{1}{x_2^-} \right)}{\left(\frac{1}{x_1^+} - \frac{1}{x_2^-} \right)} = \frac{x_1^+ (x_2^- - x_2^+)}{x_2^+ (x_2^- - x_1^+)}. \tag{154}$$

For x_2, we can use the computation we did just now:

$$\Omega(x_2^-, x_2^+) = \frac{w_0^2}{2}\left(\frac{1}{x_2^+} - \frac{1}{x_2^-}\right) = \frac{w_0^2(x_2^- - x_2^+)}{2x_2^+ x_2^-} \tag{155}$$

For x_1, it is a little bit more complicated. We find

$$\frac{d}{dx} f^{-1}(-x) = \frac{d(-y)}{dx} = -\left(\frac{dx}{dy}\right)^{-1} = -\frac{1}{f'(y)} = -\frac{1}{f'(y^-)} \tag{156}$$

So

$$\Omega = \frac{x_1^+ - x_1^-}{2}\sqrt{\frac{w_0^2}{(x_1^+)^2}\frac{1}{f'(y_1^-)}}. \tag{157}$$

We will also use

$$w_1 - w_2 \approx f^{-1}(x_1^-) = y_1^-, \quad \bar{w}_1 - \bar{w}_2 = w_0^2\left(\frac{1}{x_1^+} - \frac{1}{x_+}\right) = \frac{w_0^2(x_2^+ - x_1^+)}{x_1^+ x_2^+} \tag{158}$$

The last ingredient that we need is

$$E_S \approx \frac{c}{24\pi}\frac{2}{w_0}. \tag{159}$$

Putting everything together,

$$\begin{aligned} S &= \frac{c}{6}\log\left(\frac{w_0^2 y_1^-(x_2^+ - x_1^+)}{x_1^+ x_2^+}\frac{x_1^+(x_2^+ - x_2^-)}{x_2^+(x_1^+ - x_2^-)}\right) + \log G(\eta) \\ &- \frac{c}{6}\log\left(\frac{w_0^2(x_2^- - x_2^+)}{2x_2^+ x_2^-}\frac{x_1^+ - x_1^-}{2}\sqrt{\frac{w_0^2}{(x_1^+)^2}\frac{1}{f'(y_1^-)}}\right) \\ &= \frac{c}{6}\log\left(\frac{4}{w_0}\frac{-y_1^- x_1^+ x_2^- \sqrt{f'(y_1^-)}}{x_2^+}\frac{x_2^+ - x_1^+}{(x_1^+ - x_2^-)(x_1^+ - x_1^-)}\right) + \log G(\eta) \\ &= \frac{c}{6}\log\left(\frac{48\pi E_S}{c}\frac{-y_1^- x_1^+ x_2^- \sqrt{f'(y_1^-)}}{x_2^+}\frac{x_2^+ - x_1^+}{(x_1^+ - x_2^-)(x_1^+ - x_1^-)}\right) + \log G(\eta). \end{aligned} \tag{160}$$

9. Both endpoints to the future of the shock gives

$$w_1 = f^{-1}(x_1^-) = y_1^-, \quad \bar{w}_1 = \frac{w_0^2}{x_1^+}, \quad w_2 = f^{-1}(x_2^-) = y_2^-, \quad \bar{w}_2 = \frac{w_0^2}{x_2^+}. \tag{161}$$

We will also use

$$\Omega(x_1^+, x_1^-) = \frac{x_1^+ - x_1^-}{2}\sqrt{\frac{w_0^2}{(x_1^+)^2}\frac{1}{f'(y_1^-)}}, \quad \Omega(x_2^+, x_2^-) = \frac{x_2^+ - x_2^-}{2}\sqrt{\frac{w_0^2}{(x_2^+)^2}\frac{1}{f'(y_2^-)}}. \tag{162}$$

We also have

$$\eta = \frac{\left(f^{-1}(x_1^-) + \frac{w_0^2}{x_1^+}\right)\left(\frac{w_0^2}{x_2^+} + f^{-1}(x_2^-)\right)}{\left(f^{-1}(x_1^-) + \frac{w_0^2}{x_2^+}\right)\left(\frac{w_0^2}{x_1^+} f^{-1}(x_2^-)\right)} \approx \frac{f^{-1}(x_1^-) f^{-1}(x_2^-)}{f^{-1}(x_1^-) f^{-1}(x_2^-)} = 1. \tag{163}$$

So we can use $\log G(1) \approx 0$. The last ingredient is

$$w_1 - w_2 = f^{-1}(x_1^-) - f^{-1}(x_2^-) = y_1^- - y_2^-, \quad \bar{w}_1 - \bar{w}_2 = \frac{w_0^2(x_2^+ - x_1^+)}{x_1^+ x_2^+}. \tag{164}$$

Putting everything together yields

$$\begin{aligned} S &= \frac{c}{6}\log\left((y_1^- - y_2^-)\frac{w_0^2(x_2^+ - x_1^+)}{x_1^+ x_2^+}\right) - \frac{c}{6}\log\left(\frac{x_1^+ - x_1^-}{2}\sqrt{\frac{w_0^2}{(x_1^+)^2}\frac{1}{f'(y_1^-)}}\frac{x_2^+ - x_2^-}{2}\sqrt{\frac{w_0^2}{(x_2^+)^2}\frac{1}{f'(y_2^-)}}\right) \\ &= \frac{c}{6}\log\left(\frac{4\sqrt{f'(y_1^-)f'(y_2^-)}(y_1^- - y_2^-)(x_2^+ - x_1^+)}{(x_1^+ - x_1^-)(x_2^+ - x_2^-)}\right). \end{aligned} \tag{165}$$