Wave propagation in Schwarzschild spacetime

Consider the propagation of a massless scalar field in a spacetime of the form[1]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{1}$$

• Write down the form of the wave equation

$$\Box\Phi = \nabla^i\nabla_i\Phi = \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}g^{ij}\partial_j\Phi\right) = 0 \tag{2}$$

(keep it in a compact form).

• Observe that the angular part is the same as in the wave equation in flat space, so the angular dependence can be separated and solved by introducing the spherical harmonics $Y_{lm}(\theta,\varphi)$. Then, write the wave equation for the radial field modes $\phi_{\omega lm}(r)$ in the decomposition

$$\Phi(x^\mu) = e^{-i\omega t}Y_{lm}(\theta,\varphi)\phi_{\omega lm}(r)\,. \tag{3}$$

• We know that in flat space (which is approached as $f \to 1$) it is convenient to introduce a new radial field variable $\psi_{\omega lm}$ defined as

$$\phi_{\omega lm} = \frac{\psi_{\omega lm}}{r}\,. \tag{4}$$

In addition, for the propagation of massless excitations in the black hole background it is convenient to introduce the tortoise coordinate r_* defined as

$$dr_* = \frac{dr}{f(r)}\,. \tag{5}$$

With these changes, you must find an equation of the form

$$-\frac{\partial^2\psi_{\omega lm}(r_*)}{\partial r_*^2} = \big(\omega^2 - V_l(r)\big)\psi_{\omega lm}(r_*) \tag{6}$$

in terms of an effective potential $V_l(r)$ (which you can leave expressed in terms of r, with the understanding that r is a function of r_*).

• For the Schwarzschild spacetime $f = 1 - 2M/r$, sketch the shape of potential V_l vs. r_*/M for different values of l.

V_l is the effective radial potential that a massless scalar wave feels when propagating in this background. We have mapped this problem to one of waves in a one-dimensional potential that extends in the range $r_* \in (-\infty, +\infty)$. There are many questions that can be answered qualitatively from the shape of this potential. For instance: (i) argue that there are not any bound states of the scalar field in the black hole background. This shows (at the perturbative

[1] This is not the most general static spherical metric. Doing this exercise for the general case where $-g_{tt}(r)$ and $g^{rr}(r)$ are different functions is only a little more involved.

level) that the black hole does not admit 'scalar hair'. (ii) Waves of very low frequency coming from infinity are mostly reflected back, while waves of very high frequency will be absorbed by the black hole. Interpret these two results.

Wave propagation in AdS spacetime

Do the previous exercise for scalar wave propagation in AdS in your preferred choice of coordinates, e.g.,

$$ds^2 = -\left(\frac{r^2}{\ell^2}+1\right)dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2}+1} + r^2 d\Omega_{d-2} \tag{7}$$

$$= -\cosh^2(\rho/\ell)dt^2 + d\rho^2 + \sinh^2(\rho/\ell)\, d\Omega_{d-2} \tag{8}$$

$$= \frac{-dt^2 + \ell^2\left(d\theta^2 + \sin^2\theta\, d\Omega_{d-2}\right)}{\cos^2\theta}\,. \tag{9}$$

• Observe that now the potential is 'box-like' and admits bound states that correspond to normal modes of the field. The wave equation can actually be solved explicitly and the normal frequency spectrum can be determined analytically.