Introduction to AdS/CFT

Let me start w/ some heuristic motivation(s)

1. From black hole thermodynamics

You have learned from Roberto's lectures that b.h. behave as thermodyn objects

$$E = M \quad , \quad S \leftrightarrow A_H \quad , \quad T \leftrightarrow \kappa \quad \begin{cases} 1^{st}\text{ law} & \delta M = \frac{\kappa}{8\pi G}\delta A + \ldots \\ 2^{nd}\text{ law} & \delta A \geq 0 \text{ cls} \end{cases}$$

Bardeen, Carter, Hawking '74

- classically, b.h. have neither a temperature (absorb everything, emit nothing), nor a sizeable entropy (b.h. uniqueness)

- however, upon establishing that, QM, b.h emit thermal rad.

$$@ \quad T_H = \frac{\kappa\hbar}{2\pi} \quad \Rightarrow \quad S_{BH} = \frac{A_H}{4G_N\hbar} = \frac{A_H}{4\ell_p^2}$$

- then, expect that S_{BH} truly represents the entropy of the system for which one may expect a statistical int. as counting "b.h. microstates" (note, mostly invisible in gravity) $S = k_B \ln \Omega$
  ↳ this will be our assumption in the following

Observations:

- since $S \propto \frac{A_H}{\ell_p^2}$, it looks like the d.o.f. live on the surface of the b.h, w/ 1 d.o.f / planck-size region (~ very few) $S \propto A$ vs standard obj. w/ $S \propto V$

- on the other hand, A_H/ℓ_p^2 is a huge #. E.g. for a solar-mass b.h. w/ $R_s \sim 3$ k.m. (cf. $R_\odot \sim 700000$ km), the BH entropy is $S_{BH} = \frac{4\pi R_s^2}{4\ell_p^2} \sim \frac{30 \cdot 10^6}{(10^{-35})^2} \sim 3\cdot 10^{77} \times k_B \sim 10^{54}\, J/K \gg$ entropy sun (10^{35})

• since $S \propto A$ one would imagine that standard matter w/ $S \propto V$ could easily be more entropic than b.hs. However, this is not the case, due to gravitational collapse (generic in gravitating spt) (Penrose)

To see this, imagine a gas @ finite T (standard QFT) in a region $\mathcal{R}$ of spt of size R
We have

$$E_{gas} \sim T^4 V \sim T^4 R^3 , \qquad S_{gas} \sim T^3 V \sim T^3 R^3 \qquad \hbar = c = 1$$

dim'l analysis

we would like that $S_{gas} > S_{BH} \sim \frac{A}{G} \sim R^2/G$ (entropy of a b.h. that fills the region)

• on the other hand, the energy of the gas in region R is limited by $GE < R$ (lest it undergoes grav. collapse)

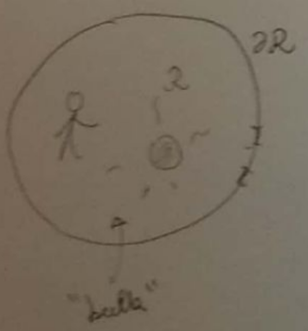
$$\Rightarrow \quad T < \left(\frac{1}{R^2 G}\right)^{1/4} \quad \Rightarrow \quad S_{gas} < \left(\frac{R^2}{G}\right)^{3/4} \sim \left(\frac{A(R)}{G}\right)^{3/4} \ll \frac{A(R)}{G}$$

⇒ the gas cannot have larger entropy than a b.h. filling the region

• the entropy is maximized for a single b.h. filling the region (as opposed to several small ones) (Pf: $S \sim \sum G M_i^2$, $R \geq G \sum M_i$ max for one M)

The implication of this simple-looking obs $S_{\text{anything in } \mathcal{R}} < S_{BH\,\mathcal{R}} = \frac{A}{4G}$ is that we can represent all that happens inside $\mathcal{R}$ by d.o.f. that live on the surface of the region ⇒

⇒ **HOLOGRAPHY** ('t Hooft '93)



fund prop of QG

→ the QG d.o.f live in fewer dims than the obs. limit indicates

→ can encode th. of gravity in $\mathcal{R}$ by th. w/o gravity living on $\partial\mathcal{R}$

→ locality → approximate, spt. → emergent

→ argument v. general, but vague
universal

2. From behaviour of large N gauge theories

- this argument (also due to 't Hooft!) suggests that certain theories w/o gravity may display an emergent grav. descr. under certain cond.

Consider $U(N)$ Yang-Mills theory (# colours) : gauge fiels $A_\mu{}^i{}_j$ (hermitian $N\times N$ matrix, indices (adj))

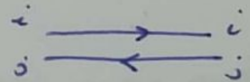
$$F_{\mu\nu}{}^i{}_j = \partial_\mu A_\nu{}^i{}_j - \partial_\nu A_\mu{}^i{}_j + g[A_\mu, A_\nu]^i{}_j \qquad \mathcal{L} = -\frac{1}{4}\operatorname{tr} F^2$$

- will be useful to use rescaled $A' \sim gA$, w/ F' ← no g expl. $\Rightarrow \mathcal{L} = -\frac{1}{4g^2}\operatorname{tr} F'^2$

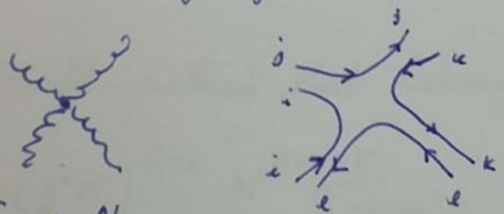
Feynman rules

$\langle A^i{}_j A^k{}_l \rangle = \langle A^i{}_j A^{\dagger k}{}_l \rangle$

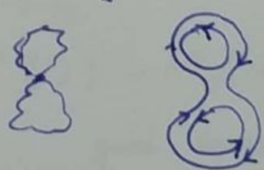
i, j  k, l $\propto g^2 \delta^{i}{}_{l}\, \delta_{j}{}^{k}$ up to μ,ν & factors of k $\frac{\eta_{\mu\nu}}{k^2}$ ...

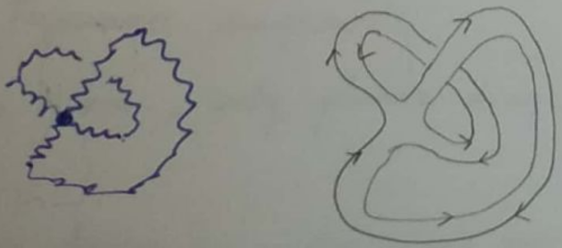
- useful to introduce double line notation 

upper index : outgoing $\rightarrow$, lower $\leftarrow$

- interactions $\frac{1}{g^2}$  $\sim \frac{1}{g^2}$ (also 3-pt vertex)

- index loop $\rightarrow \sum_i \rightarrow N$

Feynman diag will look like  $\frac{1}{g^2}(g^2)^2 N^3 \sim N g^2$ ³

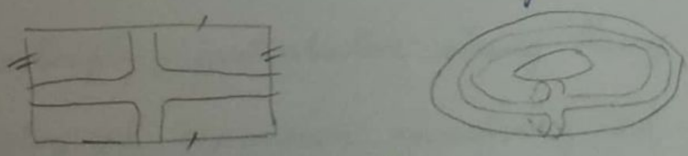


contr. to vacuum energ

$\frac{1}{g^2}(g^2)^2 N \sim Ng^2$

- the 1st diagram is called planar (can be drawn on a plane)

- the 2nd cannot be drawn on a plane, but can on a torus
  ↑ be straightened out ∋ it doesn't self-cross

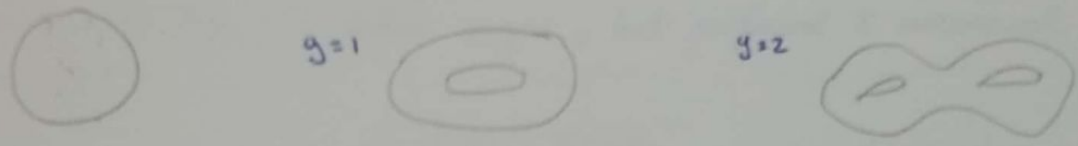


- power N = # faces after straightening

- more generally, a Feynman diag. w/ E propagators, V vertices and F loops can be straightened out on a 2d surface of genus g (# holes) w/

$$2-2g \equiv \chi = F+V-E$$

Euler character (top. invar. of 2d. surfaces $\int R\sqrt{g}$)

- 

the Feynman diagram corresp. to a partition of the surface that separates it into polygons (~ "triangulation")

- the g, N dependence of such a diagram $A \sim (g^2)^{E-V} N^F$
- since F (# loops) can be arbitrarily large → no good large N expansion
- however 't Hooft noticed that taking N large w/ $g^2 N \equiv \lambda$ fixed

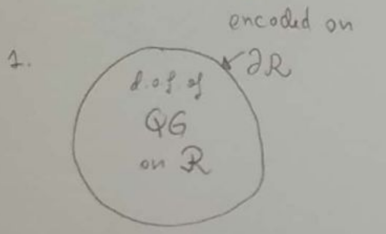
  then $A \sim (g^2 N)^{E-V} N^{\underbrace{F+V-E}_{\chi = 2-2g}}$ 't Hooft limit
- exp. in terms of topology of Feynman diagrams $N^2 f_0(\lambda) + N^0 f_1(\lambda) + \frac{1}{N^2} f_2(\lambda) + \dots$

  good large N exp. (low genus dominates)
- as $N \to \infty$, only planar diag. survive (simplification, but still have large # loops!)
- for large λ, diagrams w/ large # of loops/faces dominate as the lines become dense → smooth 2d surface
  ≈ worldsheet of a string
- interestingly, perturbative string theory is also org. into a topological expansion involving 2d surfaces

- large N gauge theory ≈> strings ? $\xrightarrow{?}$ gravity
  ('t Hooft limit)   are known to contain gravity
- this large N counting applies to any model w/ matrix-valued fields
- can also include quarks (1 fund. index) <=> hole in the Riemann surface
  (F = # index loops + # quark loops) $\chi = 2 - 2g - h$

To summarize, ∃ two ≠ rather vague, but profound & universal

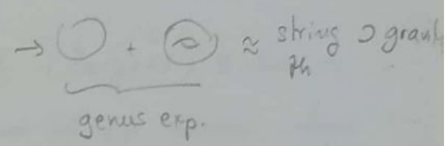
1.



gravity → no gravity

&

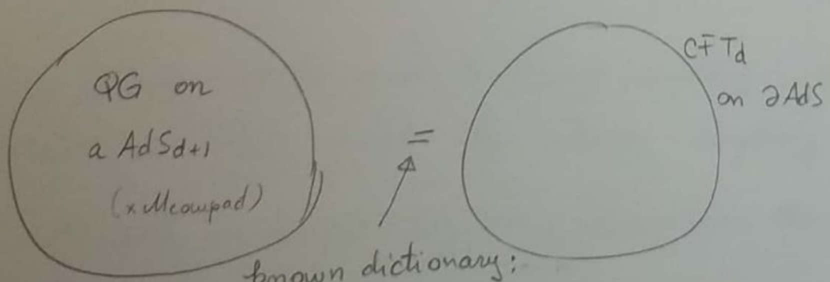
2. large N gauge th. (matrix models)



no gravity → gravity

Q: Can they be made more precise?

AdS / CFT corresp.: concrete, precise realiz. of the holographic idea for spt. that are asympt anti-de Sitter.
can only fix asympt in QG



known dictionary:
- symm, corr. f, thermodyn, entanglement ...

AdS/CFT can be discussed @ 2 levels
specif th. grav / specific CFTs (can match all ...)

1. "top-down": specific examples of AdS/CFT pairs obtained from decoupling limits of string th.
   ⊕ derivation
   only known candidate th. of QG. to date
   - whole structure of string th. (beautiful, powerful & rather rigid) stands behind

why imp?

- AdS/CFT is a strong/weak coupling duality $\mapsto$ hard to check!

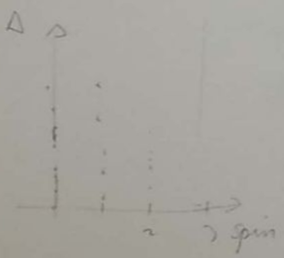
- examples are usually supersymmetric, e.g. IIB string th on $AdS_5 \times S^5$ = $\mathcal{N}=4$ $SU(N)$ SYM in $d=4$ CFT_4

  $AdS_{d+1} \times S^{9-d} \leftrightarrow CFT_d$

  but also $\exists$ AdS_3/CFT_2, AdS_4/CFT_3, AdS_7/CFT_6 etc.

- when the CFT happens to be a gauge th. (e.g. $\mathcal{N}=4$ SYM), then concrete realisation of 't Hooft's idea large N gauge th $\rightarrow$ string th.

2. "bottom-up" : $\forall$ CFT_d (notice consistent CFTs satify an ∞ # of bootstrap & other constraints)



that has a large # of d.o.f. (large N) & has a large gap in the spectrum of op. dims has a dual grav. descr. in AdS_{d+1}

part. of spin > 2 are very heavy

+ universality (AdS/CFT true indep. of string th)

- gravity in AdS has the correct structural prop. to be dual to a CFT
  - symm (& anomalies)
  - structure of corr. functions (CFT axiomatics!)
  - thermodynamics

  if a CFT w/ given prop $\exists$ (hard to establish, b/c of ∞ consist cond) then it is dual to gravity w/ prop. ...

- since corresp. relates weak & strong coupling, can use (semi-cls) gravity to learn about the strong-coupling behaviour of CFTs
  - very fruitful : AdS/CMT, holographic RG flows

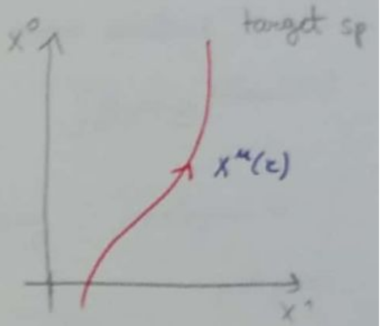
- our underst is an rich interplay of these two perspectives, w/ the concrete stringy examples guiding ppl's intuition of what can happen @ strong coupling (e.g. $\exists$ AdS/CFT) & the bottom-up perspective giving a rel. to fundamental / univ. physical phenomena & principles that do not refer to specific examples

String theory

- starting assumption: basic objects are strings, rather than pt part.

- to understand their action, useful nonetheless to start w/ a relativistic point particle analogue

  - consider a pt. particle travelling in some D-dim'l spt. w/ coord X^μ $\mu = \{0, \dots, D-1\}$



  - traces worldline $X^\mu(\tau)$ ← worldline param (arbitrary)

  - action: length spt. interval

$$ds^2 = \underbrace{G_{\mu\nu}(X)}_{\eta_{\mu\nu} \text{ for now}} \dot{X}^\mu \dot{X}^\nu \qquad \tau \; \frac{dX^\nu}{d\tau}$$

$$S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu}$$

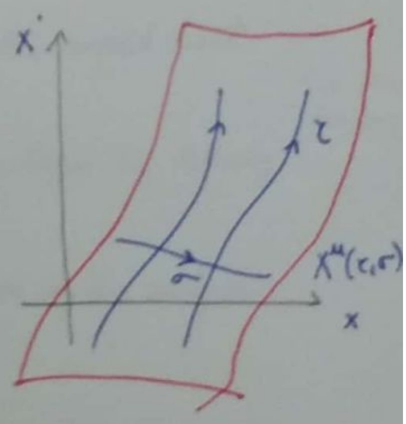
← manifest target sp Lorentz invar

← cls traj: geodesics

  - note action has reparametrization invar $\tau \to \tau' = f(\tau)$

    ⇒ constraint that momenta $p^\mu = \frac{m\dot{X}^\mu}{\sqrt{-\dot{X}^2}}$ are not indep, but satisfy $p^2 = -m^2$

- a string is an object w/ 1 spatial dim → traces a 2d worldsheet as it moves throughout spt. (target sp).



$X^\mu(\tau, \sigma)$ $\quad \mu = 0, \dots, D-1$

$\tau \in \mathbb{R}$ $\quad \sigma \in (0, \pi)$ open

$(0, 2\pi)$ closed

target sp. metric $\eta_{\mu\nu}$

· induced metric on the string = pullback target sp. metric

$$h_{ab} = \eta_{\mu\nu}\, \partial_a X^\mu \partial_b X^\nu \qquad a = 0,1 \quad \sigma^a = \{\tau, \sigma\}$$

· bosonic string action : area string w-sheet (analogy pct. part)

$$S_{NG}[X^\mu] = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det(\underbrace{\partial_a X^\mu \partial_b X_\mu}_{h_{ab}})}$$

Nambu - Goto

$$h_{ab} = \begin{pmatrix} \dot{X}_\mu \dot{X}^\mu & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix} \qquad \dot{X}^\mu = \frac{dX^\mu}{d\tau} \quad X'^\mu = \frac{dX^\mu}{d\sigma}$$

string tension $[T] \propto (\text{length})^{-2}$, $\alpha' = l_s^2$ string length

energy / unit length

· the amplitude for a string to propagate from some initial to some final state is given by the path int. $\int \mathcal{D}X^\mu e^{iS_{NG}[X^\mu]}$

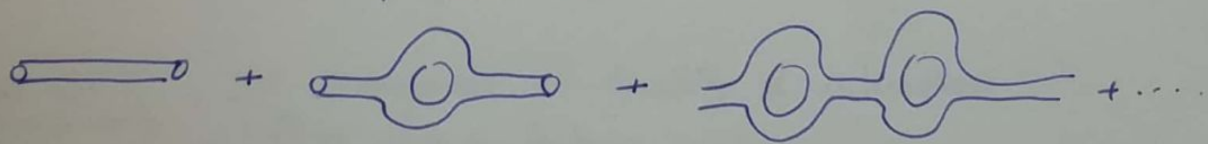
open string

closed string analogous to pct. part.

· to include perturbative interactions , in QFT we sum up Feynman diagrams

$\int \mathcal{D}X^\mu e^{-S_{NG}}$

· in string theory : Σ up smooth 2d surfaces (Euclidean for convenience)



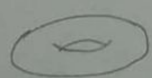
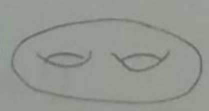
- locally, the wsheet is same as in the free case & interactions arise only from the global topology of the world sheet

str. splitting & rejoining

- one may add a weighing factor for $\neq$ topologies by letting

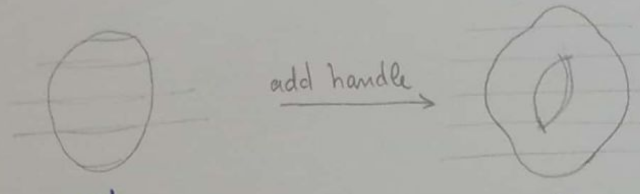
$$S_{string} = S_{NG} + \lambda \chi \leftarrow \text{Euler character of the surface}$$

· for e.g. vacuum processes we have

$\mathcal{Z}_{string} = e^{+2\lambda} \times$  $+$  $+ e^{2\lambda}$  $+ \dots$

same genus exp. we found in large N gauge th!

• e^{λ} → interpreted as ampl. for emitting a string ~ g_s
string coupling



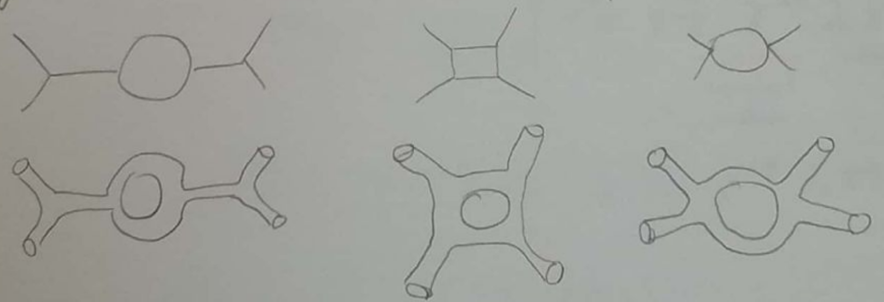
$e^{-2\lambda}$ string emerges/nucleates from ∅ gets reabs. into vac.

$e^{0} = e^{-2\lambda} \cdot e^{2\lambda}$ $\begin{cases} e^{\lambda} & \text{emit} \\ e^{\lambda} & \text{reabsorb} \end{cases}$

string emerges from vac spits out a string, reabs. it & disappears into vac

• thus string perturbation th. → classification of euclidean 2d surfaces
simple → genus g

• in a certain limit, one may think of string "Feynman" diagrams as just fattened QFT diag → note, however, that # of string diag @ given loop order << # QFT Feynman diagrams, e.g.

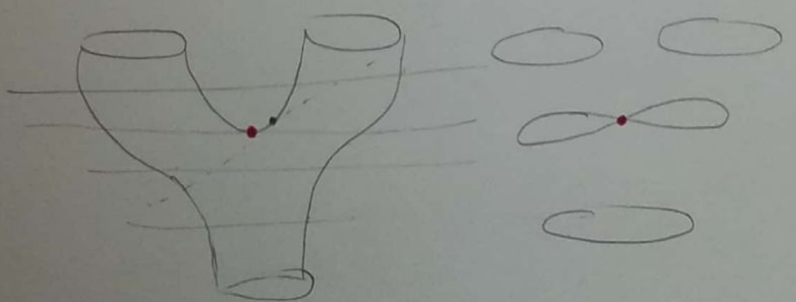


all ≠

same topology ≠ ε param.

• in QFT, UV divergences ← locality of interactions

• in string theory, interaction is smoothened out in spt → no UV divs



but also ...

• thus, I sketched wh. str. pert. th. is org. in a genus exp. Let me now argue it contains gravity (more work!)
(& gauge th.)

Spectrum of the bosonic string

- action $\quad S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma \sqrt{-\det(\partial_a X^\mu \partial_b X_\mu)}$

  ← D scalars from w-sheet perspect.

- symmetries : D-dim'l Poincaré $\quad X'^\mu(\tau,\sigma) = \underbrace{\Lambda^\mu{}_\nu}_{\text{Lorentz}} X^\nu(\tau,\sigma) + \underbrace{a^\mu}_{\text{transl.}}$

  - reparametrization invar $(\tau,\sigma) \to (\tau'(\tau,\sigma), \sigma'(\tau,\sigma))$

    ⇝ constraints

- if cls → quantum th → path int. over X^μ $\quad \int \mathcal{D}X^\mu e^{-S}$

  hard to deal w/ the square root NG action

- a classically equiv. form of the string action : Polyakov action

$$S_P[X^\mu, \gamma_{ab}] = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-\gamma}\, \underbrace{\gamma^{ab}}_{\text{w-sheet metric}} \underbrace{\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}}_{h_{ab}}$$

- γ_{ab} e.o.m. $\quad T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu = 0$

  solve for γ_{ab} & plug in to get S_{NG}.

  $T_{ab}=0$

- symmetries S_P : Poincaré, reparam & Weyl : $\gamma_{ab} \to e^{2\xi(\tau,\sigma)}\gamma_{ab}$

  arbitrary rescalings of the metric keeping angles fixed

- can use these to gauge-fix $\gamma_{ab} = \eta_{ab}$ (✓ cls th.)

- residual freedom : conformal transf.

- we basically get D free bosons, but w/ some constraints ($T_{ab}=0$)

  CFT$_2$

$$\dot{X}^\mu X'_\mu = 0 \qquad \dot{X}^2 + X'^2 = 0$$

useful to introduce lightcone coord on w-sheet $\sigma^\pm = \tau \pm \sigma$ (LM / RM)

then constraints : $(\partial_+ X)^2 = \partial_+ X^\mu \partial_+ X_\mu = 0 \qquad (\partial_- X)^2 = \partial_- X^\mu \partial_- X_\mu = 0$

· bnd. cond. the variation of the action is

2π for closed

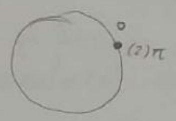
$$\delta S = \frac{1}{2\pi\alpha'}\left[\int d\tau d\sigma \left(\eta^{ab}\partial_a\partial_b X_\mu\right)\delta X^\mu - \int d\tau\, \partial_\sigma X_\mu \delta X^\mu \Big|_{\sigma=0}^{\pi}\right]$$

$\partial_{\sigma^+}\partial_{\sigma^-}X^\mu = 0$

for the action to be minimized, need not only $\Box X^\mu = (\partial_\sigma^2 - \partial_\tau^2)X^\mu = 0$

but also the appropriate bnd. conditions. Several possibilities

- closed strings : periodic & continuous

  $X^\mu(\tau,0) = X^\mu(\tau,2\pi)$, $\partial_\sigma X^\mu(\tau,0) = \partial_\sigma X^\mu(\tau,2\pi)$

- open strings : Neumann bnd. cond $\partial_\sigma X^\mu(\tau,\sigma)\Big|_{\sigma=0}^{\pi} = 0$

  endpt. can sit anywhere in tg. sp.

  - Dirichlet bnd. cond $\delta X^\mu(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0$

    endpts. are @ fixed positions in target space

· sols to the e.o.m. can be expanded in modes

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \qquad \text{left \& right - moving}$$

$$X^\mu_{\underset{R}{L}}(\sigma^\pm) = \frac{1}{2}\underbrace{x^\mu}_{\text{C.M. position}} + \frac{1}{2}\alpha'\underbrace{p^\mu}_{\text{C.M. momentum}}\sigma^\pm + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^\pm}$$

closed str.

$\left.\begin{matrix}\alpha_n^\mu \\ \tilde{\alpha}_n^\mu\end{matrix}\right\}$ oscillators

reality cond $(\alpha_n^\mu)^+ = \alpha_{-n}^\mu$

· p^μ = momentum in target sp. (symm. assoc. w/ $X^\mu \to X^\mu + a^\mu$)

$\approx J_0^\mu$

$p^2 \to -\text{mass}^2$ of the string $\approx$ looks pointlike from far away

mass of the assoc. particles (spectrum)

· open strings : X^μ_R det. by X^μ_L via bnd. cond. $\partial_\sigma X^\mu|_{0,\pi} = 0$

(Neumann) $\Rightarrow \tilde{\alpha}_n^\mu = \alpha_n^\mu$

+ constraints $(\partial_+ X)^2 = (\partial_- X)^2 = 0$.

- quantization: promote α_n^μ, $\tilde{\alpha}_n^\mu$ to operators w/ canonical comm. rels

$\Leftarrow \quad [X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = i\delta^\mu{}_\nu\, \delta(\sigma-\sigma') \qquad [X^\mu, X^\nu] = [\Pi_\mu, \Pi_\nu] = 0$

& require that physical states be annih. by the constraints (careful!)

$[X^\mu, p_\nu] = i\delta^\mu{}_\nu \qquad [\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\, \eta^{\mu\nu} \delta_{m+n,0}$

2 ∞ towers of osc.

$\alpha_n^\dagger = \alpha_{-n} \qquad \underset{n>0}{\alpha_{-n}}$: creation ops, $\underset{n>0}{\alpha_n}$: annih. ops.

(standard. cr-annih. ops $\quad a_n^\dagger = a_{-n} = \dfrac{\alpha_{-n}}{\sqrt{n}}$, $n>0$ & a_n)

- build Fock space as in standard QFT

$\alpha_n^\mu |0\rangle = \tilde{\alpha}_n^\mu |0\rangle = 0 \quad , \forall\ n>0$

$\underset{\text{operator}}{\hat{p}^\mu} |0\rangle = \underset{\text{eigenval}}{p^\mu} |0\rangle$ b/c this is vacuum st. of a single string, (not vacuum QFT)
which can carry momentum $\mapsto$ write $|0;p^\mu\rangle$

- excited states $\alpha_{-1}^{\mu_1} \dots \alpha_{-1}^{\mu_n} \alpha_{-2}^{\nu_1} \dots \alpha_{-2}^{\nu_p} \dots\ |0,p\rangle$

classified into representations of the target sp. Lorentz gp.

- the constraints are

$$\underbrace{\partial_+ X^\mu}_{} \cdot \partial_+ X_\mu = \frac{\alpha'}{2} \sum_{m,p} \alpha_m^\mu (\alpha_p)_\mu\, e^{-i(\underbrace{m+p}_{n})\sigma^+} = \frac{\alpha'}{2} \sum_n \overbrace{\sum_m \alpha_m^\mu (\alpha_{n-m})_\mu}^{2L_n} e^{-in\sigma^+}$$

$\frac{1}{2}\alpha' \underset{\alpha_0^\mu}{p^\mu} + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \alpha_n^\mu e^{-in\sigma^+}$

$\mapsto$ need to (normal) order

- cannot impose $L_n |phys\rangle = 0$, $\forall n$, b/c $[L_m, L_n] \sim \delta_{m+n,0}$

- only impose this for $n>0$.

- we thus require that $L_n |phys\rangle = 0 \quad \forall\ n>0$

$(L_0 - a)|phys\rangle = 0$ normal ordering const. $(a=1)$

+ RM (if closed string) α_0

$L_0 = \frac{\alpha'}{4} p^2 + \sum_{n>0} \alpha_{-n}^\mu \alpha_{n,\mu} \qquad \tilde{L}_0 = \dots$

almost a # op

- the operator $\sum_n \alpha_{-n}^{\mu} \alpha_{n,\mu}$ counts # α_{-n}'s w/ a factor of n in front when acting on the Fock space states → level $N = \sum_n n N_n$

- remembering that $p^2 = -M^2$ (mass² in tg. sp), the phys st. cond. for $L_0, \tilde{L}_0$ lead to the mass-shell cond $\boxed{M^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N}-a)}$

det. spacetime spectrum of the string

- note in the closed string $N = \tilde{N}$ (level matching)
- consistency : $a=1$ & $D=26$ $\left(a = \frac{D-2}{24}\right)$

Lessons : 1. From a target space persp., the various excit. of the string ⟷ ∞ tower of massive (higher spin) particles w/ M^2 as above → spacing $\propto 1/\alpha'$

(they all fit into massless/massive reps of Lorentz gp)

2. low-lying spectrum

- closed string : $N = \tilde{N} = 0$ tachyon $\left(M^2 = -\frac{4}{\alpha'}\right)$
unstable vacuum ↦ ignore (susy)

- $N = \tilde{N} = 1$ $\alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0, p\rangle$
w/ $M^2 = 0$

in terms of $SO(1, D-1)$ reps : $g_{\mu\nu}$ ($G_{\mu\nu}$, $h_{\mu\nu}$) , $B_{\mu\nu}$, Φ

$g_{\mu\nu}$	$B_{\mu\nu}$	Φ
symm. traceless	antisymm	dilaton
spin 2	B-field	$\int d\tau d\sigma\, \phi R \sqrt{\gamma}$
graviton !	gauge field	$\simeq \phi \chi$
↓	coupling to string	natural $g_s \simeq e^{\phi_0}$
$\int d\tau d\sigma\, G_{\mu\nu} \partial_a X^\mu \partial^a X^\nu$	$q \int d\tau\, A_\mu \dot{X}^\mu$	exp. value of field
	$\int d\tau d\sigma\, B_{\mu\nu} \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu$	

⇒ (closed) string theory contains gravity! ← massless spin 2

- ! only consistent QG we know of !

- massless string mode prop. on fixed Minkowski backgnd ↦ target sp. geom. becomes dynamical!

- @ low energies ($E \ll \frac{1}{\sqrt{\alpha'}}$) the massive string modes can be neglected (into)

⇒ effective action for the massless ones $S_{EH} + S_B + S_\phi$ coupled

$$S = \frac{1}{2\kappa_0^2} \int d^Dx \sqrt{-G}\, e^{-2\phi} \left[R - \frac{1}{12} \underbrace{H_{\mu\nu\rho} H^{\mu\nu\rho}}_{\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}} + 4 \partial_\mu \phi \partial^\mu \phi + \underbrace{\mathcal{O}(\alpha')}_{\text{higher order corr. suppressed by } \alpha'} \right]$$

↦ standard Einstein gravity @ low eng. (+matter)

- open string (low-lying spectrum) $M^2 = \frac{1}{\alpha'}(N-1)$

  - $N = 0$ tachyon (ignore)

  - $N = 1$ $\alpha_{-1}^\mu |0, p\rangle$ → $\underbrace{\text{photon } A_\mu}_{\text{contains gauge fields}}$ $\int_{\partial\Sigma} d\tau\, A_\mu \frac{dx^\mu}{d\tau}$ coupling

    spin 1. $\neq \int d^Dx \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$

- if we consider instead superstrings (add fermions on w.-sheet)

  - (+) can project out the tachyons (stable vacuum)
  - makes sense in $D = 10$
  - additional massless fields : p-forms "RR" $\underbrace{C_{\mu_1 \dots \mu_p}}_{\text{antisymm.}}$ IIB even / IIA odd + gaugino fermions
  - has target sp. supersymmetry (low-eng eff. action highly constrained)

e.g. $$S_{IIB} = \frac{1}{(2\pi)^7 \alpha'^4} \int d^{10}x \sqrt{G} \left[e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H_{\mu\nu\rho}^2\right) - \frac{1}{2}\left(F_\mu^2 + \frac{1}{3!} \tilde{F}_{\mu\nu\rho}^2 + \underbrace{\frac{\tilde{F}^2_{\mu\nu\rho\sigma\tau}}{5!}}_{\text{careful}}\right)\right.$$

$$+ \text{ CS terms } + \text{ fermions}$$

- effective 10d Newton's const. $G_{10} \sim \alpha'^4 g_s^2$ (ℓ_P^8) $g_s = \langle e^\phi \rangle$ vev dilaton

pert. string exp. → two scales : $\ell_s = \sqrt{\alpha'}$ effects of massive string modes

$\sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g(\alpha'/R^2)$ (R char. scale backgnd) $\ell_P = g_s^{1/4} \ell_s$ QG effects. Note $\ell_P \ll \ell_s$ if g_s small

E: QG / string / sugra

D - branes

- all discussion so far was perturbative (in g_s)
- will now discuss obj that will turn out to be non-perturbative
- they lead to huge progress in string theory
  - 1^{st} microscopic explanation of black hole entropy $S = \ln \Omega$
  - the discovery of AdS/CFT
- importantly, $\exists$ two ways to think about them

1. remember open strings can have either Neumann ($\partial_\sigma X^\mu = 0$) or Dirichlet ($\delta X^\mu = 0$) bnd. cond. @ their endpts

Dirichlet membrane

- a Dp-brane (p+1 dim'l hypersurface, p # spatial dir) is specified by: Neumann bnd. cond along the hypersurf.

$$\partial_\sigma X^\mu |_{\sigma=0,\pi} = 0 \qquad \mu = \{0, \dots, p\}$$

- Dirichlet b.c. in the transverse directions

$$\delta X^m |_{\sigma=0,\pi} = 0 \qquad m = \{p+1, \dots, D-1\}$$

- a priori, just a hyperplane where open strings can end
- however, it should be considered a fully dynamical object
  - remember a massless open string excit is a gauge field, which couples as $\int A_\mu \dot{X}^\mu d\tau$ to the string endpts
  - $\exists$ a symmetry of string th. (T-duality) that turns $N \leftrightarrow D$ bnd. cond.

$\parallel$ dir $\qquad\qquad$ $\perp$ dir

$$N: \int A_\mu \partial_t X^\mu \qquad\qquad D: \int \phi_m \partial_n X^m$$

- from the hyperplane's point of view we now have a $p+1$ dim'l vector A_μ & $D-p-1$ scalars ϕ^m ($\Leftarrow$ from A_m before T-duality)

- the scalars ϕ^m (massless open string excit) should be thought of as the transverse displacement of the Dp-brane

  - this is similar to the closed string case, where a massless closed string excit ($h_{\mu\nu}\,\alpha_{-1}^{(\mu}\tilde{\alpha}_{-1}^{\nu)}|0,p\rangle$) about flat target sp. corresponds to a fluctuation of the tg. sp. geom

  - here, a massless open string state (ϕ^m) $\rightarrow$ fluctuation of the hypersurf. $\Rightarrow$ D-branes are dynamical objects
    (makes sense since in the grav $\nexists$ rigid objects)

- low-eng eff. action $\sim \frac{1}{g_s \alpha'^{\frac{p-3}{2}}} \int d^{p+1}x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu \phi^m)^2\right)$

- generaliz: interesting to add labels to the open string endpts
  non-dyn. d.o.f.

  i ——— j Chan-Paton factors $\approx$ QCD string, these would be quarks

  - general open string state $|N, p, ij\rangle$ (level N $(=1)$) $i,j = \{1 \ldots N\}$ #!

  - we now have N^2 massless gauge fields A_μ^{ij} (& N^2 scalars ϕ^m_{ij} if Dirichlet) (Hermitean matrices) w/ a $U(N)$ symm acting on these, as can be seen from low-eng. eff. action

all N:
$$S = \frac{1}{g_{YM}^2}\int d^D x \left[-\frac{1}{4}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\right] \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

  - invar under $U(N)$ gauge symm $A_\mu \rightarrow U A_\mu U^{-1} + i\,\partial_\mu U\, U^{-1}$

  - so a global wsheet symm $\mapsto$ gauge tg. sp. symm

  - in presence of both N & D bnd. cond w/ N Chan-Paton factors, the low-eng. eff action (no dep. on x^m, slow on x^m) of the corresp N D-branes is

$$S = \frac{1}{g_{YM}^2}\int d^{p+1}x \left[-\frac{1}{4}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2}\mathrm{tr}(\mathcal{D}_\mu \phi^m \mathcal{D}^\mu \phi_m) + \mathrm{tr}[\phi^m, \phi^n]^2\right]$$

· the coupling of the D-branes to the target sp. fields is

$$S_{DBI} = -\frac{1}{\alpha'^{\frac{p+1}{2}}}\int d^{p+1}x \left\{ e^{-\phi}\, tr\left(\sqrt{-\det(G_{\mu\nu}+B_{\mu\nu}+2\pi\alpha' F_{\mu\nu})}\right) + O[X^{\mu},X^{\nu}]^2 \right\}$$

induced metric & B-field

world vol + corr.

$G_{\mu\nu} = \partial_\mu X^M \partial_\nu X^N G_{MN}$ etc.

· physical tension is $\frac{1}{g_s \alpha'^{\frac{p+1}{2}}}$ => non-perturbative object (very heavy for $g_s \ll 1$)

· this action reduces to the one above for $E \ll \frac{1}{\sqrt{\alpha'}}$

· in superstring theory Dp-branes can be shown to carry RR charge $\mu_p \int d^{p+1}x\, C^{(p+1)}_{\mu_1 \dots \mu_p}$ & to preserve half supersymmetry

(important for their identif @ strong coupling)

- low-energy effective action = susy version of U(N) Yang-Mills

- $g^2_{YM} = g_s \alpha'^{\frac{p-3}{2}}$, For $p=3$ the SYM theory is conformally inv.

g_{YM} is an exactly marginal coupling

To sum up: D-branes are non-pert. obj, must be incl in open str. th via T-duality, 1/2 susic & charged under the RR (p+1)-form fields.

- described by SYM @ low eng (tractable @ small 't Hooft couplig $\lambda = N g^2_{YM}$

(2.) D_p-branes: solitonic sols of the low-eng eff. action

- extended in p+1 directions
- charged under RR fields
- 1/2 susic if extremal

basically, extended generaliz. of the Reissner-Nordström black hole

· the only terms in the eff action we will need are (sol $B=0$, etc)

RR field strength

$$S = \frac{1}{(2\pi)^7 \alpha'^4}\int d^{10}x \sqrt{g}\left[e^{-2\phi}(R + 4(\partial\phi)^2) - \frac{2}{(8-p)!} F^2_{p+2} \right]$$

· look for a sol'n w/ $\mathbb{R}^{1,p}$ Poincaré symm & spherically symm. in $\perp$ 9-p directions

p extremal

• the sol'n for the (string frame) metric & dilaton is

$$ds^2 = +\frac{1}{\sqrt{H(r)}}\left(-dt^2 + dx_i^2 + \dots + dx_p^2\right) + \sqrt{H(r)}\left(dr^2 + r^2 d\Omega^2_{8-p}\right)$$

$$e^\phi = g_s H(r)^{\frac{3-p}{4}} \qquad H(r) = 1 + \frac{\# g_s N \alpha'^{\frac{7-p}{2}}}{r^{7-p}}$$

also $C_{0\dots p} = H^{-1}(r)$ charge N under F_{p+2} $\quad \int_{S^{8-p}} *F_{p+2} = N$

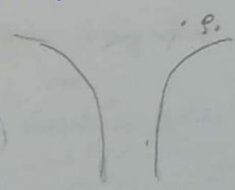
• to better underst. the analogy w/ RN black holes, remember they have

$$ds^2 = -\left(1 - \frac{2GM}{\rho} + \frac{GQ^2}{\rho^2}\right)dt^2 + \frac{d\rho^2}{1 - \frac{2GM}{\rho} + \frac{GQ^2}{\rho^2}} + \rho^2 d\Omega_2^2 \qquad A_0 = \frac{Q}{\rho}$$

• horizons @ $\rho_\pm = GM \pm \sqrt{(GM)^2 - GQ^2}$ extremal for $M = Q/\sqrt{G}$

Hawking temp. vanishes.

• spt. geom develops a very long throat



$$\text{dist}(\underset{\text{fixed}}{\rho_0}, \rho_+) = \int_{\rho_+}^{\rho_0} \frac{d\rho}{\sqrt{1 - \frac{2GM}{\rho} + \frac{GQ^2}{\rho^2}}} = 2\ln\left(\sqrt{\rho - \rho_+} + \sqrt{\rho - \rho_-}\right)\Big|_{\rho_+}^{\rho_0}$$

$$\simeq -\ln(\rho_+ - \rho_-) \to \infty \text{ as } \rho_- \to \rho_+$$

• if the b.h. is extremal $f(\rho) = \left(1 - \frac{Q\sqrt{G}}{\rho}\right)^2 = \left(\frac{r}{\underbrace{r + Q\sqrt{G}}_{\rho}}\right)^2 = \frac{1}{\left(1 + \frac{Q\sqrt{G}}{r}\right)^2}$

horizon is @ $r=0$ in this coord.

analogous $H(r)$ D-brane

• Dp-branes have a horizon @ $r=0$. (more generally, @ analogue of ρ_+ if non-extremal)

• for $p \le 6$ $\exists$ also curvature sing @ ρ_-, extremal: null sing. on horizon if $p \ne 3$

↳ smooth if $p=3$.

• for $p < 3$, dilaton blows up @ horizon

• $p=3$ extremal sol'n smooth & can be ext. past hor.

also $e^\phi = \text{const}$ → will concentrate on this from now on

- before moving on to the deriv. of the AdS/CFT corresp, let us comment on the range of validity of this sol'n:

- note we used classical supergravity, which is valid provided:
  (stringy corr. are negligible)
  - curvatures are smaller than string scale

    size S^5 in n.h. lim. given by (= horizon radius in original coord).

    $(\neq g_s N \alpha'^2)^{\frac{1}{4}} \gg \sqrt{\alpha'} \Rightarrow \boxed{g_s N \gg 1}$ *

    (*) Note from p.d.v. of the gauge th. descr. this is the regime of very large 't Hooft coupling (v. hard to calc. in f.t.)

  - also need to suppress string loops. $\Rightarrow g_s \ll 1$ $(\Rightarrow l_p \ll l_s)$

  - thus, need $\boxed{1 \ll g_s N < N}$ N large
    $g_s N$ large.

alternatively
- more gen $l_p^8 = g_s^2 \alpha'^4 \ll r_+^8 = (g_s N \alpha'^2)^2 \Rightarrow N \gg 1$ QG suppressed, but not necessarily stringy

  $\sim g_s N \gg 1$ sugra valid

  $\sim g_s N < 1$ highly stringy, but not QG

- for non-conformal branes ($p \neq 3$), sugra description only valid for a certain range of r. (nicely fits w/ $r \Leftrightarrow$ energy scale, but do this later)

Summary

- D-branes are well-descr by gauge th for $E \ll \frac{1}{\sqrt{\alpha'}}$, w/ pert. calc. valid for $g_s N \ll 1$

  - well-described by sugra for $g_s N \gg 1$, w/ $N \gg 1$

- the identif. of the 2 descr. (highly) non-triv!) follows from BPS, charges & duality ← more precisely? string th. considerations

Deriving the AdS_5/CFT_4 corresp.

- consider now the coupled brane/bulk system

- consider IIB string th. in flat sp. + $N \parallel$ D3 branes (sitting very close to each other) along $x^0, \dots, x^3$

- string th. in this backgnd. contains 2 types of excit: $\begin{cases} \text{closed strings (excit. empty sp.)} \\ \text{open strings (excit. D-branes)} \end{cases}$

- if we consider very low energies $E \ll \frac{1}{\sqrt{\alpha'}}$, then only massless states will be excited.

- eff. Lagr. for them (Wilsonian)

$$S = \underbrace{S_{bulk}}_{\substack{\text{10d sugra} \\ \text{(+ higher deriv)} \\ \text{that take into account effect of} \\ \text{int. out massive string modes)}}} + \underbrace{S_{brane}}_{\substack{\mathcal{N}=4 \text{ SYM} \\ \text{+ higher deriv. corr.}}} + \underbrace{S_{int}}_{\substack{\text{int. b/w. brane} \\ \text{\& bulk.} \\ (\propto G_N^{\#})}}$$

$\frac{1}{G_N}\int R\sqrt{g}\, d^{10}x + \dots$ $\qquad \int F^2 d^4x + \dots$

e.g. $\frac{1}{g_s\alpha'^2}\sqrt{\det(G_{ab} + B_{ab} + \alpha' F_{ab})}$

positive powers

. Let $g = \eta + \kappa h$, $\kappa \sim \sqrt{G_N}$

$\ni \frac{1}{\kappa^2} R\sqrt{g} \to (\partial h)^2$ to lowest order

then a term $h^\ell A^k \partial^n$ comes w/ coeff $\frac{1}{g_s\alpha'^2}(\kappa h)^\ell (\alpha' A)^k$

- to undust. better what happens in the low eng. limit, $E \ll \frac{1}{\sqrt{\alpha'}}$, one may keep E fixed & send $\alpha' \to 0$ w/ g_s, N fixed.

open string picture ①

- then $G_N = g_s^2\alpha'^4 \to 0 \quad \Rightarrow S_{int} = 0$, ~~$S_{higher\ deriv}$~~ $= 0$.
- $S_{higher\ derivs} \to 0$ (both on brane & in the bulk)

two decoupled systems

$$\Rightarrow S = \underbrace{S_{bulk}}_{\substack{\hookrightarrow \text{free gravity in flat sp.} \\ \text{(very low eng. lim)}}} + \underbrace{S_{brane}}_{\mathcal{N}=4 \text{ SYM}}$$

- now, consider the same limit, viewing the Dp-branes as sugra sols ②

$$ds^2 = \frac{1}{\sqrt{H(r)}}\left(-dt^2 + dx_1^2 + \dots + dx_3^2\right) + \sqrt{H(r)}\left(dr^2 + r^2 d\Omega_5^2\right) \qquad e^\phi = g_s$$

$$F_5 = (1+*)\, dt \wedge dx^1 \dots \wedge dx^3 \wedge d(H^{-1}) \qquad H(r) = 1 + \frac{4\pi g_s N \alpha'^2}{r^4}$$

- note that, due to the warp factor, the energy of an obj. measured from ∞ & by an obs P @ r are rel by $E_\infty = E_p \underbrace{\sqrt{-g_{tt}(r)}}_{\text{redshift factor}}$

$E_1\sqrt{-g_{tt}(r_1)} = E_2\sqrt{-g_{tt}(r_2)}$

(careful string vs Einst. unit. for $p \neq 3$)

$$E_\infty = \frac{E(r)}{H^{1/4}} \to 0 \text{ as } r \to 0$$

- => same object, if brought closer & closer to $r=0$, appears to have lower & lower eng from the p.d.v. of ∞ ($E(r)$ could be rest mass.)

- taking the low energy limit mentioned above → 2 types of low-eng. excitations from the p.d.v. of an obs @ ∞
  - long wavelength modes prop. in the bulk (near ∞)
  - modes very close to $r=0$ (~~inside the throat~~). There, the geom. is

$$\left(H(r) \propto \frac{R^4}{r^4}, \quad w/ \; R^4 = 4\pi g_s N \alpha'^2 \right) \qquad \sqrt{4\pi g_s N}\,\alpha'$$

$$ds^2 = \underbrace{\frac{r^2}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + R^2 \frac{dr^2}{r^2}}_{AdS_5} \times \underbrace{+ R^2 d\Omega_5^2}_{S^5} \quad w/ \text{ radius } R$$

- these two types of modes completely decouple in the low eng. limit.
  - for modes in the throat, hard to climb out.
    - consider throwing up a part from some $r_0 \ll 1$
    - cons.ⁿ eng. along geodesic $E = -\underset{\frac{1}{\sqrt{H}}}{g_{tt}} \frac{dt}{d\tau}$ $\quad + \underset{\frac{1}{\sqrt{H}}}{g_{tt}}\left(\frac{dt}{d\tau}\right)^2 + \underset{\sqrt{H}}{g_{rr}}\left(\frac{dr}{d\tau}\right)^2 = -1$

      $\sqrt{H}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{g_{tt}} E^2 = -1$

      $E^2 = \frac{1}{\sqrt{H}} + \left(\frac{dr}{d\tau}\right)^2$ p.b.f E_∞
  - conversely, ~~for~~ modes w/ λ large / ω small do not "see" the ~~D-branes~~ throat region ($\sim$ size R) (the abs. cross section $\sigma \sim \omega^3 R^8$ for ω small).

- we see that from either p.d.v. of the D-branes, we obtain 2 decoupled systems one of which is free sugra in flat sp. => natural to identify the other two.

$r \simeq 0$ region of the D3 soln.

$$\boxed{\mathcal{N}=4 \text{ SYM th. on } \mathbb{R}^{1,3} \underset{\forall N, g_s}{=} \text{IIB } \cancel{\text{sugra th.}} \text{ string th. on } AdS_5 \times S^5}$$

- to see we get full type IIB: we want to ~~#~~ take the $\alpha' \to 0$ limit ∃ energies in the throat are fixed in string units (consider arbitrary excited (stringy) st. from m.h. p.d.v). This => $E_p \sqrt{\alpha'}$ fixed

- $E_p\sqrt{\alpha'}$ fixed $\rightarrow$

$$E_\infty = H^{-1/4}E_p \approx \frac{r}{(g_sN)^{1/4}\sqrt{\alpha'}} \frac{E_{fixed}}{\sqrt{\alpha'}}$$

- since $\underline{E_\infty}$ is the energy measured in field theory $\xrightarrow{\text{want}}$ fixed $\rightarrow$ $r \propto \alpha'$ as $\alpha' \to 0$.

or, take $\alpha' \to 0$ w/ $u \equiv \frac{r}{\alpha'}$ fixed. (also, gauge boson mass arg $m \sim \frac{r}{\alpha'}$ fixed undust!)

(letting $r = \alpha' u$, then notice $ds^2 = \alpha' \left(\frac{u^2}{(4\pi g_sN)^{1/2}}(-dt^2 \ldots) + \frac{du^2}{u^2}(4\pi g_sN)^{\frac{1}{2}} + (4\pi g_sN)^{\frac{1}{2}} \ldots \right)$

- notice string th side appears to have both g_s & α', while YM side only $g_{YM}^2 = g_s$
  - however, α' actually drops out from string side; in part, plugging in $r = u\alpha'$ into $ds^2 \propto \alpha'$ $\rightarrow$ overall factor that cancels from phys. quantities (how about finite size?)
  - we may work in terms of a rescaled metric $d\tilde{s}^2 \rightarrow ds^2 = R^2 d\tilde{s}^2$ (effectively, this sets $R = 1$)

    you are in 't Hooft limit

    $$\Rightarrow \underbrace{\alpha' \sim \frac{1}{\sqrt{g_sN}}}_{\text{controls } \alpha' \text{ corr (stringy)}} \quad \& \quad G_N = g_s^2 \alpha'^4 \sim \underbrace{\frac{1}{N^2}}_{\text{controls QG corr.}}$$

    ($\forall$ quantity calc in pure grav is indep of g_sN)

- as already discussed, QG effects are suppressed for N large (if g_s large & N small w/ g_sN large, the D-branes are tiny)
  - sugra valid for $R \gg \sqrt{\alpha'}$ $\Rightarrow$ $\underbrace{g_sN}_{g_{YM}^2 N}$ large

    't Hooft coupling of SYM th (pert)

- notice the range of $g_sN = g_{YM}^2 N$ for which the sugra / SYM descr. are valid (& controllable) are perfectly incompatible
  - this is actually good, as it would have been strange if 2 theories that a priori look so different are the same.
  - thus AdS/CFT is a strong/weak coupling $\boxed{\text{duality}}$ ← expl.

{ • hard to prove • useful: solve cft strong coupling ... when one side is weakly coupled (easy to compute), the other one is str, so h... }

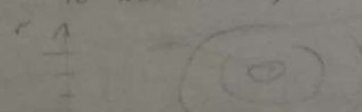
- why is this argument not a proof?
  - because we do not know how to treat the string th. side non-perturbatively (not even in principle).
- can distinguish $\neq$ "strengths" of the proposed corresp.
  - weakest: SYM is described by grav. @ large N & $g_s N = \lambda$, but the full string th $\neq$ field theory (e.g., $1/\lambda$ corr would not agree) Tests w/o susy
  - medium, SYM = str. th. $\forall\, \lambda = g_s N$, but only for $N \to \infty$ (so, $1/N$ corr. may not agree)
  - strong form, & most interesting: true @ $\forall\, g_s, N$ can use the CFT ($N=4$ SYM) ~~to de~~ @ finite N to define what we mean by QG in a AdS.
    non-pert. str th

Current status of tests strength? (w/o susy?)

- if the CFT is used to define the gravity theory, then the AdS radial direction & gravity are emergent from the CFT pd.v. What is their meaning in the CFT?
  - traditionally, r has been associated w/ the energy scale in the dual CFT.

$$ds^2 = r^2(-dt^2 + d\vec{x}^2) + \frac{dr^2}{r^2}$$

to see this, note that



$$E_{FT} = \sqrt{-g_{tt}}\, E_{proper}(r) \to \infty \text{ as } r \to \infty$$

$$\underbrace{\text{Size}_{FT}}_{\Delta x} = \frac{1}{r}\,(\text{proper size}) \to 0 \text{ as } r \to \infty$$

- thus, short distances / high energies in FT $\leftrightarrow$ large r in gravity
  known as UV/IR corresp
  - more recently, spt architecture (connectedness, geom) as well as the emergence of gravity have been connected to the structure of entanglement in the CFT (will discuss in last lecture)

★ Generalizations

- the above decoupling arg. can be run also for other brane systems in string / M-theory (11d th ← strong coupling lim of type IIA)

(⇒) ∴ other proposed dualities b/w CFT_d & string / M-th (large N)

- D1-D5 ↔ IIB on $AdS_3 \times S^3 \times T^4$
- M5 / M2 ↔ $AdS_7 \times S^4$ & $AdS_4 \times S^7$ backgnds in M-th

- there also ∃ decoupling limits that yield non-conformal (& non-local) th
  - Dp for $p \neq 3$ complicated backgnd
  - NS5 branes (LST) & 7d AF spt. w/ linear dilaton $\times S^3$
  - Dp-branes in non-trivial B-field backgnd. → non-commutative FT ↔ string th. fuzzy backgrounds

- all these are explicit examples of holographic dualities b/w gravitational theories (string th on some backgnd) & non-gravitational ones living in one (non-compact) dim. less.
- all non-trivial realiz of the holographic principle within the str. th. context

★ How to test this duality? → need a dictionary!

- since the theories are supposed to be the same,

$$\boxed{Z_{gravity}[\text{bnd cond}] = Z_{FT}[\text{operator sources}]}$$

can fluctuate inside — careful

in a th. of grav, doesn't make sense to fix backgnd (but can fix it asympt) Σ all geom. w/ fixed asy. all crap. inside Top. changes allowed

- will concentrate on the AdS/CFT examples from now on
  - best understood
  - universal flavour to the corresp, partly due to CFTs having a univ, axiomatic defn that is also valid @ strong coupling (encompass all examples w/ a univ. set. of rules)

The AdS / CFT dictionary

AdS	CFT	
symm	✓	same
states spectrum	✓	
fields	operators	

- you have already learned that the (global) conf. gp in d dimensions is $SO(d,2)$
- one can discuss the symm. of AdS @ 2 levels
  - symm. of the vacuum sol^n (isometries of empty AdS)
  - symm. of the theory → asympt. symm. of AdS.

I will start w/ the first of these

Anti-de Sitter spacetimes

i) AdS_{d+1} maximally symm. spt (max'l # KV $\frac{(d+1)(d+2)}{2}$) of const. negative curvature

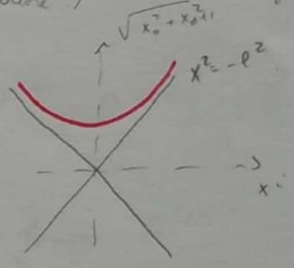
$$R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$ same curv. everywhere

as computing using the expl. KV SO(d,2) algebra

- isometry gp $SO(d,2)$ (for Lorenzian AdS_{d+1}) → same as "Lorentz" rot. in $\mathbb{R}^{d,2}$

, will now show global AdS can be obtained from a manif $SO(d,2)$ inv. constr, which is not w/ its own right

- simplest construction : embed into $\mathbb{R}^{d,2}$, w/ metric



$$ds^2 = \tilde{\eta}_{MN}\, dX^M dX^N \qquad \tilde{\eta} = \begin{pmatrix} - & & & \\ & + & & \\ & & + & \\ & & & + \; - \end{pmatrix}$$

(manifest $SO(d,2)$)

& restrict to hyperboloid $\tilde{\eta}_{MN} X^M X^N = -\ell^2$ $SO(d,2)$ invar. constraint

- a metric on AdS_{d+1} can be obtained by writing down an explicit coord system that solves the constraint, e.g.

$$\left.\begin{aligned} X^0 &= \ell \cosh\rho \cos\tau \\ X^{d+1} &= \ell \cosh\rho \sin\tau \\ X^i &= \ell \sinh\rho\, \Omega_i\,, \quad w/ \sum_{i=1}^{d} \Omega_i^2 = 1 \end{aligned}\right\} \text{ global coord}$$

$$ds^2 = \ell^2\left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{d-1}^2\right)$$

decompactify   unit S^{d-1} $SO(d)$ manifest.

$\rho \in [0,\infty)$, $\tau \in [0, 2\pi)$ covers hyperboloid once

before decomp

- manifest $SO(2) \times SO(d) \subset SO(d,2)$

letting $ch\,\rho = \frac{1}{\cos\theta}$ ... the metric conf to that on a cyl the conformal generators can be easily identified from the embedding space using

$J_{AB} = -i\left(X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}\right)$ actual gen

· the quadratic Casimir $\frac{1}{2} J_{AB} J^{AB}$ is identified w/ the AdS Laplacian (only isometry-invar 2nd order diff. op)

· acting on e.g. a scalar, we have $\frac{1}{2} J_{AB} J^{AB} \phi = [-X^2 \Box_X + X\cdot\partial_X (d + X\cdot\partial_X)]\phi$
$\Box_X$: $\mathbb{R}^{d,2}$ Laplacian resp.

$\mathbb{R}^{d,2}$ can be foliated by AdS_{d+1} slices w/ $\neq \ell$ $ds^2_M = -d\ell^2 + \ell^2 ds^2_{AdS_{d+1}}$

the Laplacian is $\Box_X = -\partial_\ell^2 - \frac{d+1}{\ell}\partial_\ell + \Box_{AdS}$
ℓ not resc.

$\frac{1}{2} J_{AB} J^{AB} \phi = [\ell^2(-\partial_\ell^2 - \frac{d+1}{\ell}\partial_\ell + \Box_{AdS}) + \ell\partial_\ell(d + \ell\partial_\ell)]\phi = \ell^2 \Box_{AdS}\phi$

- in for a free scalar $\Box_{AdS}\phi = m^2\phi$

in CFT, Casimir $\Delta(\Delta - d)\phi = m^2\ell^2\phi$.
for a highest-weight repr.

mass of field in spt $\leftrightarrow$ conf. dim. in CFT $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2\ell^2}$

· similar constr. for euclidean global AdS : start from $\mathbb{R}^{1,d+1}$

& let $X^0 = \ell \cosh\rho \cosh\tau$, etc.

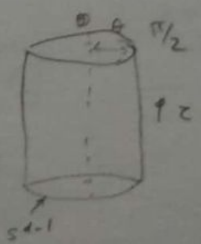
· AdS spt is non-compact → how can we repr. it?

- conformal compactif : given (M, g) non-compact, choose $\tilde{g} = \Omega^2 g$ (conf fact Ω)
$\exists\ (M, \tilde{g})$ can be embedded into a compact domain $\tilde{M}$ of some manif.
$\partial\tilde{M}$ = conf. bnd

- in practice, choose coord on M w/ finite ranges

$ds^2 = \ell^2(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega^2_{d-1})$ $\cosh\rho = \frac{1}{\cos\theta}$ $\rho \in (0,\infty)$, $\theta \in (0, \frac{\pi}{2})$

$= \frac{\ell^2}{\cos^2\theta}(-d\tau^2 + \underbrace{d\theta^2 + \sin^2\theta\, d\Omega^2_{d-1}}_{\text{metric on half } S^d \simeq \text{disk}})$



- $d\tilde{s}^2 = \cos^2\theta\, ds^2 \simeq \text{disk} \times \mathbb{R}_+$

⇒ Penrose diag of AdS_{d+1} = ∞ solid cylinder w/ bnd $S^{d-1} \times \mathbb{R}_+$

· since $\tilde{g} = \Omega^2 g$, the causal structure w.r.t. $\tilde{g}$ is the same as w.r.t g (even though distances are heavily distorted)

Exercise 1: show that lightrays shot from the center of AdS take a finite coord time to reach the bnd.

(you can assume all motion takes place in the radial plane)

· thus, if we impose reflecting bnd. cond for lightrays @ the AdS bnd, they come back in a finite coord time, $\pi \cdot \ell$

· this is despite the fact that the dist. along a spacelike geod ($\tau = \Omega = 0$) is infinite : $\ell \int_0^{\pi/2} \frac{d\theta}{\cos\theta}$

AdS acts as a confining box for massive part

Exercise 2 show that a massive part.'s geodesics never reach the bnd.

(assume, for simplicity, that motion is just in radial plane)

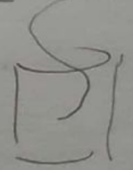
∂_τ = Killing vect → cons. quantity $E = -\xi^\mu g_{\mu\nu} \frac{dx^\nu}{dT} = -g_{\tau\tau}\frac{d\tau}{dT} = \frac{\ell^2}{\cos^2\theta}\frac{d\tau}{dT}$ (T proper time)

$$ds^2 = -dT^2 = \frac{\ell^2}{\cos^2\theta}(d\theta^2 - d\tau^2) \Rightarrow \frac{\ell^2}{\cos^2\theta}\left(\frac{d\theta}{dT}\right)^2 = \frac{\ell^2}{\cos^2\theta}\left(\frac{E\cos^2\theta}{\ell^2}\right)^2 - 1$$

$$= \frac{E^2\cos^2\theta}{\ell^2} - 1 > 0$$

⇒ $\cos\theta$ cannot become too small

$$\cos^2\theta_{min} = \frac{\ell^2}{E^2}$$



light : impose refl bnd. cond (don't have to) ; not imposing bhs evaporate

⇒ AdS behaves as a confining box for massive part. independently of their initial energy.

· massless fields ~~can~~ escape, but we usually impose refl. bnd. cond @ ∂AdS so that they don't.

· ∃ other useful coord systems in AdS, e.g. Poincaré

· cover patch AdS

embedding $X^0 = \frac{\ell^2 + x_\mu x^\mu + z^2}{2z}$, $X^i = \ell \frac{x^i}{z}$ ($i = 1 \ldots d-1$), $X^d = \frac{\ell^2 - x_\mu x^\mu - z^2}{2z}$, $X^{d+1} = \ell\frac{t}{z}$

$$ds^2 = \frac{\ell^2}{z^2}(\underbrace{\eta_{\mu\nu}dx^\mu dx^\nu}_{R^{d-1,1}} + dz^2)$$

· make Poincaré dilatation symm. manifest $x^\mu \to \lambda x^\mu$, $z \to \lambda z$

conf. bnd @ $z = 0$

Asymptotic symm

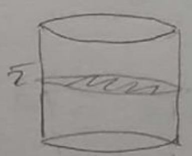
- now, isometries vac. $\neq$ symm. of the theory
- for CFT_d, $d>2$ the two are the same
- however, in 2d CFTs, the symm of the theory are Vir × Virasoro

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\,\delta_{m+n,0} \qquad + RM\ (\tilde{L}_m)$$

$x^+ \to e^{imx^+}\cdot \varepsilon$ quantum anomaly $\quad T^\mu{}_\mu = -\frac{c}{24\pi} R[\gamma]$ non-triv metric

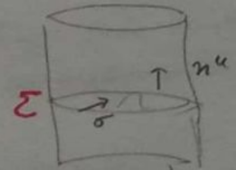
– the vacuum state only preserves the global $SO(2,2)$ subgp of $Vir_L \times Vir_R$ w/ gen $L_{0,\pm1}$ & $\tilde{L}_{0,\pm1}$

The symm of the theory are repr. in gravity as asymptotic symm.

- usually, one assoc. cont symm w/ Noether currents, cons. on-shell $\partial_\mu J^\mu = 0$
- the cons. charges are given by  $Q = \int_\Sigma d^{d-1}x\, J^t$
- the Noether current stays cons if one adds a bnd term $J^\mu \to J^\mu + \partial_\nu k^{\mu\nu}$ (anti.)

which contributes to the charge as $\Delta Q = \int_{\partial\Sigma} d^{d-2}x\, k^{rt}$

- if the param of the symm is local $\varepsilon(x^\mu)$ (gauge theories), then the standard assoc Noether current vanishes on-shell up to a bnd. term that can be systematically constructed
- thus, in gauge th (incl. gravity: $\xi^\mu(x)$), the symm. are asymptotic b/c. the cons. charges take the form



$$Q_\xi = \int_{\partial\Sigma} d^{d-2}x \sqrt{\sigma}\, \underbrace{k_\xi^{\mu\nu}}_{\text{dep. on action \& } \xi} n_\mu \sigma_\nu \qquad \text{\& vanish if } \xi^\mu\big|_{\partial\Sigma} = 0$$

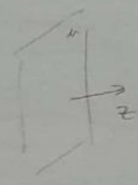
· of course, ∀ theory is defined by an action & a set of bnd. cond. $\mathcal{C}$.
on a manif w/ a bnd.

- an allowed diffeomorphism is such that $\underbrace{\mathcal{L}_\xi g_{\mu\nu}}_{\mathcal{C}} \in \mathcal{C}$.
gauge transf

- a trivial diffeo is such that $Q_\xi \equiv 0$ ∀ $g_{\mu\nu} \subset \mathcal{C}$ b/c
ξ falls off too fast near infinity

The asympt symm of the theory are def as $\dfrac{\text{Allowed diffeos}}{\text{Trivial diffeos}}$

& they corresp to true symm. of the theory



Example 1. Maxwell field in AdS_{d+1}

- bnd. cond : A_μ fixed as $z=0$ $(A_\mu = 0)$ as $z \to 0$
- allowed gauge transf : $\varepsilon(x^\mu, z)$ ∋ $\partial_\mu \varepsilon = 0$ @ z_0

$$\varepsilon(x^\mu, z) = \varepsilon_0 + \underbrace{z^* \varepsilon^{(1)}(x^\mu)}_{O(z)\ \text{gt}} + \dots$$
trivial $\qquad Q_\xi \simeq \int \varepsilon(x,z) F^{rt}$

⇒ asympt symm : constant
phase rot on the ∂AdS ⇒ global symm!

dict.: A_μ in bulk ⟷ J^μ on the bn
gauge field $\qquad$ CFT current → global symm.

Example 2 : asympt symm. of AdS_3 $\qquad x^\pm = \varphi \pm t$

- go to FG. gauge $\quad ds^2 = \underbrace{\ell^2}_{AdS_3\ \text{radius}} \dfrac{dz^2}{z^2} + \underbrace{\dfrac{\overbrace{d\varphi^2 - dt^2}^{dx^+ dx^-}}{z^2}}_{\text{bnd. cond}} + \underbrace{\mathcal{L}(x^+) dx_+^2 + \bar{\mathcal{L}}(x^-) dx_-^2}_{\text{universal}}$

+ subleading (non-universal) (dep. on matter fields)

· the bulk diffeos that preserve this form of the metric must asympt become CKV of the bnd metric, $\eta_{\alpha\beta}$
$\chi_L(x^+)$, $\chi_R(x^-)$

asympt AdS_3 spt.

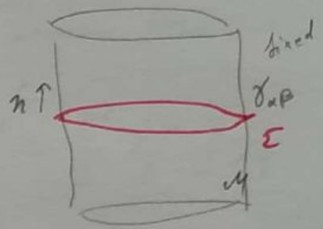
• thus, the allowed diffeos take the form

$$\xi_L = \chi_L(x^+)\partial_+ + \frac{1}{2}\chi_L'(x^+)\, z\partial_z - \frac{\ell^2 z^2}{2}\chi_L''(x^+)\partial_- + \dots$$

$$\xi_R = \chi_R(x^-)\partial_- + \frac{1}{2}\chi_R'(x^-)\, z\partial_z - \frac{\ell^2 z^2}{2}\chi_R''(x^-)\partial_+ + \dots$$

• now, to decide what are trivial & non-triv. diffeos, need to compute cons. charges. A convenient formula is based on the Brown-York stress tensor

$$T_{\alpha\beta} = -\frac{2}{\sqrt{\gamma}}\frac{\delta S^{\text{on-shell}}[\gamma]}{\delta\gamma^{\alpha\beta}} \qquad S = \overbrace{S_{EH}}^{\int R\sqrt{G}} + \overbrace{S_{GH}}^{\frac{1}{8\pi G}\int_{\partial M} K} + S_{ct}[\gamma]$$



$$= -\frac{1}{8\pi G}(\underbrace{K_{\alpha\beta}}_{\frac{1}{2}n^\lambda\partial_\lambda\gamma_{\alpha\beta}} - K\gamma_{\alpha\beta}) + ct$$

$$Q = \int_{\partial\Sigma} d^{d-2}x\,\sqrt{\sigma}\, n^\alpha T_{\alpha\beta}\xi^\beta$$

• for the aAdS$_3$ metric above

$$K_{\alpha\beta} = -\frac{z}{2\ell}\partial_z\gamma_{\alpha\beta} = \frac{1}{\ell}\left(\frac{\eta_{\alpha\beta}}{z^2} + O(z^2)\right)$$

$$K = \gamma^{\alpha\beta}K_{\alpha\beta} = z^2\Big(\overbrace{\eta^{\alpha\beta}}^{\begin{pmatrix}0&1\\1&0\end{pmatrix}} - z^2\begin{pmatrix}\bar{\mathcal{L}} & \cdot\\ \cdot & \mathcal{L}\end{pmatrix}\Big)\left(\frac{\eta_{\alpha\beta}}{z^2} + O(z^2)\right) = \frac{2}{\ell} + O(z^4)$$

$$T_{\alpha\beta} = -\frac{1}{8\pi G\ell}\left(\frac{\eta_{\alpha\beta}}{z^2} - 2\left(\frac{\eta_{\alpha\beta}}{z^2} + \begin{pmatrix}\mathcal{L} & \\ & \bar{\mathcal{L}}\end{pmatrix}\right)\right) + O(z^2) = \frac{1}{8\pi G\ell}\begin{pmatrix}2\mathcal{L} & \frac{1}{z^2}\\ \frac{1}{z^2} & 2\bar{\mathcal{L}}\end{pmatrix} + O(z^2)$$

divergent!

$$Q \sim \int_{z\sim\varepsilon} d\varphi\; z\; T_{t\beta}\,\xi^\beta$$

need to add counterterm to make it finite $\quad S_{ct} = -\frac{1}{8\pi G\ell}\int d^2x\,\sqrt{\gamma}$

$$\Delta T_{\alpha\beta} = -\frac{2}{\sqrt{\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{\alpha\beta}} = -\frac{1}{8\pi G\ell}\gamma_{\alpha\beta}$$

$$T_{\alpha\beta}^{\text{ren}} = \frac{1}{8\pi G\ell}\begin{pmatrix}\mathcal{L} & 0\\ 0 & \bar{\mathcal{L}}\end{pmatrix}$$

& the charges are $Q_{\xi_L} = \frac{1}{8\pi G l} \int_0^{2\pi} d\varphi \, \mathcal{L}(x^+) \chi_L(x^+)$ finite!

$Q_{\xi R} = -\frac{1}{8\pi G l} \int d\varphi \, \bar{\mathcal{L}}(x^-) \chi_R(x^-)$

... are trivial.

Symm algebra: $\{ Q_{\eta_L}, Q_\xi \} = \delta_\xi Q_\eta = \frac{1}{8\pi G l} \int_0^{2\pi} d\varphi \, \delta_\xi \mathcal{L}(x^+) \chi_\eta(x^+)$

$$\delta_\xi \mathcal{L} = 2\mathcal{L} \chi'_\xi(x^+) + \chi_{\xi_L}(x^+) \mathcal{L}'(x^+) - \underbrace{\frac{l^2}{2} \chi'''_{\xi_L}(x^+)}_{\text{inhom. term}}.$$

$$\{ Q_{\eta_L}, Q_{\xi_L} \} = \frac{1}{8\pi G l} \int_0^{2\pi} d\varphi \left[2\mathcal{L} \chi'_\xi + \chi_\xi \mathcal{L}' - \frac{l^2}{2} \chi'''_\xi \right] \chi_\eta$$

$$= \frac{1}{8\pi G l} \int_0^{2\pi} d\varphi \, \mathcal{L}(\chi'_\xi \chi_\eta - \chi'_\eta \chi_\xi) - \frac{l}{16\pi G} \int_0^{2\pi} d\varphi \, \chi'''_\xi \chi_\eta$$

$Q[\eta, \xi]_{L.B}$.

$\chi_\eta = e^{imx^+} \qquad \chi_\xi = e^{inx^+}$

$$\{ Q_m, Q_n \} = \frac{1}{8\pi G l} \int_0^{2\pi} d\varphi \, \mathcal{L}(x^+) \, i(n-m) \, e^{i(m+n)x^+} - \frac{l}{16\pi G} \int d\varphi \, (in)^3 e^{i(m+n)x}$$

$$= -i(m-n) Q_{m+n} - \frac{l}{16\pi G} (in)^3 \cdot 2\pi \, \delta_{m+n,0}.$$

$[\ ,\ ] = i\{\ ,\ \}_{PB} \qquad Q$

$$[L_m, L_n] = (m-n) L_{m+n} + \underbrace{\frac{l}{8G}}_{c/12} m^3 \delta_{m+n,0}$$

Vir w/ $c = \frac{3l}{2G}$

quantum symm alg CFT

cls CFT calc.

Holographic correlation functions

- The AdS/CFT dictionary is an equality of partition functions

$$Z_{bulk}[\text{bnd. cond}] = Z_{bnd}[\text{sources}]$$

- would like to underst how it works for a field in the bulk
  → free scalar $\Box\phi = m^2\phi$

- Poincaré coord $ds^2 = \underset{\text{AdS rad.}}{\ell^2} \dfrac{\eta_{\mu\nu}dx^\mu dx^\nu + dz^2}{z^2}$

- the e.o.m. takes the form $\left(z^2\partial_z^2 + z^2\Box_\eta + (1-d)z\partial_z - m^2\ell^2\right)\phi(z,x^\mu) = 0$

- 2^{nd} order eqn → 2 indep sols.

  assuming $\phi \sim z^\Delta$ as $z \to 0$, we find $\Delta(\Delta-1) + (1-d)\Delta - m^2\ell^2 = 0$

$$\Rightarrow \quad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2\ell^2} \qquad \begin{array}{l} \Delta_+ = \Delta \\ \Delta_- = d-\Delta \end{array}$$

  thus, the near-bnd expansion of the scalar field takes the form

$$\phi(z,x^\mu) = \underbrace{\phi_0(x^\mu)}_{\text{non-norm}} z^{d-\Delta} + \underbrace{\ldots}_{\text{terms} \sim \partial^n \text{deriv of } \phi_0} + \underbrace{\phi_\Delta(x^\mu)}_{\text{norm}} z^\Delta + \underbrace{\ldots}_{\text{deriv}}$$

$\int \frac{dz}{z^{d+1}} z^2 (\partial_z\phi_0)^2$

- since $\phi_0 z^{d-\Delta}$ is non-normalizable → ϕ_0 needs to be fixed as a bnd. cond. At the level of the full bulk field $\phi(x^\mu, z)$, we require that

$$\phi(x^\mu, z)\Big|_{z=\varepsilon} = \varepsilon^{d-\Delta}\phi_0(x^\mu) \quad \text{as } \varepsilon \to 0$$

ϕ_0 - infinity

  "renormalized bnd cond" (finite as $z \to 0$)

- ϕ_Δ is normalizable & thus allowed to fluctuate.

- note that under a rescaling of the CFT coord, $x^\mu \to \lambda x^\mu = x'^\mu$, which corresp to the AdS isometry $x^\mu \to \lambda x^\mu$, $z \to \lambda z$, ϕ is invariant, while

$$\phi_0(x')\lambda^{d-\Delta} = \phi_0(x) \qquad \phi_\Delta(x')\lambda^\Delta = \phi_\Delta(x)$$

  same transf as source for same transf as operator of dim Δ

- thus, more concretely, in this case the AdS/CFT dict. can be written as

$$Z_{bulk}\left[\phi(z,x^\mu)\big|_{z\to 0} = z^{d-\Delta}\phi_0(x)\right] = \left\langle e^{-\int d^dx\, \phi_0(x)\, \mathcal{O}_\Delta(x)} \right\rangle_{CFT}$$

generating f. of gauge-invar ops. in the CFT $e^{-W_{CFT}[\phi]}$

- arbitrary corr. f. can be obtained by differentiating $W_{CFT}[\phi_0(x)]$ w.r.t. $\phi_0(x)$

- true in general (bulk ~ string th) but we will concentrate on the sugra limit $N\to\infty$, λ large, in which $\underset{\text{saddle pt}}{Z_{bulk}[\phi_0] \approx} e^{-S_{sugra}[\phi_0]}$

bulk on-shell action w/ bnd. cond ϕ_0

Two-point function

- only quadratic part of the action needed $S = \int d^{d+1}x \sqrt{g}\, \frac{1}{2}\left((\partial\phi)^2 + m^2\phi^2\right)$

- the soln is as before $\phi(z,x) = \phi_0 z^{d-\Delta} + \dots + \phi_\Delta z^\Delta + \dots$

- $\phi_0(x^\mu)$, $\phi_\Delta(x^\mu)$ are indep from the p.d.v of the near bnd. analysis; however, they get related by requiring that the soln not blow up ~~in~~ as $z\to\infty$

- solving the wave eqn in momentum sp $\phi(z,x) = e^{ip_\mu x^\mu} f(z)$

sols $z^{\frac{d}{2}} I_\nu(pz)$ & $z^{\frac{d}{2}} K_\nu(pz) = z^{\frac{d}{2}}\left(I_\nu(pz) - I_{-\nu}(pz)\right)$

$p = |p|$ $\qquad I_\nu = \left(\frac{pz}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}$

$$= z^{\frac{d}{2}+\nu}\left(\frac{p}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} - z^{\frac{d}{2}-\nu}\left(\frac{p}{2}\right)^{-\nu}\frac{1}{\Gamma(1-\nu)}$$

$$\Rightarrow \phi_\Delta = -\left(\frac{p}{2}\right)^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(\nu+1)}\phi_0$$

- let us now evaluate the on-shell action
  - rather $\delta S_{on\text{-}shell}[\phi_0] \leftrightarrow \langle \mathcal{O}_\Delta(x^\mu)\rangle_{\phi_0}\, \delta\phi_0$

$$\delta S = \int d^{d+1}x\sqrt{g}\,\underbrace{(m^2\phi - \Box\phi)}_{0}\,\delta\phi + \int d^dx \sqrt{\gamma}\; \underbrace{n^\mu}_{}\,\partial_\mu\phi\,\delta\phi$$

normal to the bnd @ $\underbrace{z=\varepsilon}_{\text{regulator IR/UV}}$ $\quad -\frac{z}{\ell}\partial_z$

$$\delta S_{on\text{-}sh} = -\int d^dx\ \frac{z}{\ell}\,\partial_z\phi\,\delta\phi\cdot\left(\frac{\ell}{z}\right)^d = -\int_{z=\varepsilon} d^dx\left(\frac{\ell}{z}\right)^{d-1}\Big[\ \phi_0(d-\Delta)\,z^{d-\Delta-1}+\dots$$

$$\dots+\Delta\,\phi_\Delta\,z^{\Delta-1}+\dots\Big]\cdot\left(\delta\phi_0\ z^{d-\Delta}+\dots\ +\delta\phi_\Delta\,z^{\Delta}+\dots\right)$$

Note:

- coeff $\phi_0\,\delta\phi_0$ divergent $\dfrac{\varepsilon^{2d-2\Delta-1}}{\varepsilon^{d-1}} = \varepsilon^{d-2\Delta}$ div for $\Delta > \frac{d}{2}$
  - subleading divs $\Box^n\phi_0\,\delta\Box^n\phi_0$
- finite term (ε^0) $\quad -\left(\phi_0(d-\Delta)\,\delta\phi_\Delta+\Delta\,\phi_\Delta\,\delta\phi_0\right)$

does not have a good variational principle, as only ϕ_0 should be held fixed (so, $\delta S_{onsh}\propto\delta\phi_0$)

- to obtain meaningful & finite results we should - as in QFT - add bnd. counterterms to remove the divs. In AdS/CFT, the counterterms are local, Lorentz invar, & depend only intrinsically on the bnd geometry ($\phi|_\varepsilon$, $\gamma_{\mu\nu}|_\varepsilon$ etc.).

→ holographic renormalization

- easy to see the leading divergence can be absorbed via the counterterm $\quad S_{ct} = \dfrac{c}{\ell}\displaystyle\int d^dx\,\sqrt{\gamma}\,\phi^2(x,z)\Big|_{z=\varepsilon}$ choosing c appropriately

$$\delta S_{ct} = \frac{2c}{\ell}\int_{z=\varepsilon} d^dx\left(\frac{\ell}{z}\right)^d\left(\phi_0\,z^{d-\Delta}+\dots\right)\left(\delta\phi_0\,z^{d-\Delta}+\dots\right)$$

$$-\phi_0\,\delta\phi_0\,(d-\Delta)+2c\,\phi_0\,\delta\phi_0=0\quad\Rightarrow\quad c=\frac{d-\Delta}{2}$$

- this ct. also modifies the coeff. of the subleading terms (div. or not) & in part, the finite term coeff. becomes

$$-\left(\phi_0(\cancel{d}-\Delta)\,\delta\phi_\Delta+\Delta\,\phi_\Delta\,\delta\phi_0\right)+(d-\Delta)\left(\cancel{\phi_0}\,\delta\phi_\Delta+\phi_\Delta\,\delta\phi_0\right)$$

$$=(d-2\Delta)\,\phi_\Delta\,\delta\phi_0 \qquad \text{problematic term cancels!}$$

& $\delta S\propto\delta\phi_0$

- the subleading divs can be cancelled by ct $\int d^dx\,\sqrt{\gamma}\,\phi\,\Box^n\phi$ which do not affect the finite term.

· Thus $\dfrac{\delta S_{\text{on-shell}}[\phi_0]}{\delta\phi_0} = (2\Delta - d)\,\phi_\Delta = \langle O_\Delta \rangle_{\phi_0}$

⇒ the coeff ϕ_Δ in the near-bnd. expansion of the scalar repr. the expectation value of the dual bnd. operator

· to obtain the 2pt.-f $\langle O_\Delta O_\Delta \rangle = \dfrac{\delta \langle O_\Delta \rangle}{\delta \phi_0} \propto p^{2\nu}$ in momentum space $\langle O(p) O(-p) \rangle$

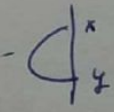
· Fourier-transforming $\int d^d p \; e^{ip\cdot x}\, p^{2\nu} \sim \dfrac{1}{(x_1 - x_2)^{2\Delta}}$ $\quad \Delta = \frac{d}{2} + \nu$

exactly the 2p.f of an operator of dim. Δ.

Comments

· here, free scalar; higher spin fields very similar → higher spin ops in CFT

· for massless gauge fields in the bulk (A_μ, $g_{\mu\nu}$) → cons. currents on bnd; cons. eqn. obtained from near-bnd.-analysis (hologr. Ward identities, including anomalies)



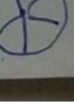
· interesting to note $\dfrac{1}{|x-y|^{2\Delta}}$ is $e^{-\Delta \ell}$ ℓ ren. geod. length b/w the pts x, y on the bnd also holds more gen. in non-vac. spt.

· to compute higher-pt. functions, need sol'n @ higher order in $\phi_{[0]}$ → include bulk interactions. e.g. for $\frac{\lambda}{n+1}\phi^{n+1}$, the modified e.o.m. is $(\Box - m^2)\phi = \lambda \phi^n$

solve pert $\phi = \phi^{[0]} + \lambda \phi^{[1]} + \lambda^2 \phi^{[2]} + \dots$ $\qquad (\Box - m^2)\phi^{[0]} = 0$
$\qquad (\Box - m^2)\phi^{[1]} = \phi_{[0]}^n$ etc

⇒ $\phi^{[1]}_{(y)} = \int d^{d+1}y' \; G_{bb}(y|y')\, \phi^n_{[0]}(y')$ $\qquad \phi_{[0]}(y') = \int d^d x \; G_{b\partial}(y, x)\, \phi_0(x)$

as $y \to$ bnd norm.-soln $\psi^{[1]}_{\Delta (x)} = \int d^{d+1}y' \; K_\Delta(x, y') \prod_{i=1}^{n} \int d^d x_i \; K_\Delta(y', x_i)\, \phi^{[0]}_{(x_i)}$ witten diag.



Theories at finite temperature

- so far, we have mostly studied the AdS/CFT dictionary about the AdS vacuum, where the vacuum isometries determine much of the structure (of e.g. correlation functions)

- we would now like to turn to the study of theories @ finite temperature. As you know, the thermal part. f. of a th on a sp. manif M w/ Hilbert sp $\mathcal{H}$ (& Hamilt. H) can be computed via the euclidean path int.

$$Z_\beta = \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H} = \int_{M_4 \times S^1_\beta} \mathcal{D}\phi\, e^{-S_E(\phi)} \qquad \beta = \frac{1}{T}$$

- the gravity dual of a thermal CFT state is given, using the master AdS/CFT dictionary, by

$$Z_{grav}[\underbrace{M_{d-1} \times S^1_\beta}_{\text{conf. bnd of AdS}}] = Z^\beta_{CFT}$$

{ can of course also add sources for ops, but let's ignore this for now

- thus, we are supposed to compute the gravity path integral : Σ over all eucl metrics that asympt. take the form (in AdS_5) $M_{d-1} = S^3$

$$ds^2 \xrightarrow[r\to\infty]{} \frac{r^2}{\ell^2} dt_E^2 + \frac{\ell^2}{r^2} dr^2 + r^2 d\Omega_3^2 \quad \text{w/} \quad t_E \sim t_E + \beta$$

- in the semicls limit $Z_{grav} \simeq e^{-S_E^{cl}}$ ← on-shell action for the classical sols to the e.o.m. (at least approx.) w/ the prescribed bnd. cond. If there are several such sols

$$Z_{grav}(\beta) = e^{-S_E^{(1)}}\underbrace{(1+\dots)}_{\text{pert. corr}} + e^{-S_E^{(2)}}(1+\dots)$$

the one w/ the smallest eucl. action dominates $Z_{grav}(\beta)$.

· in aAdS$_5$ w/ bnd $S^3 \times S^1$, there are 3 sols that satisfy the above bnd. cond: large b.h, small b.hs & thermal AdS

The Schwarzschild-AdS black hole

· eucl. b.h. sols w/ given bnd. cond.

$$ds^2 = f\,dt_E^2 + \frac{dr^2}{f} + r^2 d\Omega_3^2 \qquad f = 1 + \frac{r^2}{\ell^2} - \frac{\mu}{r^2} \qquad \ell = \text{AdS length}$$

$\mu \propto$ mass $\quad \frac{16 G_5}{3\pi^2} M$

$$t_E \sim t_E + \beta$$

(ratio of asympt S^1 & S^3 r, as only ratio makes ~~sense~~ in conf. inv th.)

· horizon $f(r_+) = 0 \Rightarrow r_+^2 = \frac{\ell^2}{2}\left(-1 + \sqrt{1 + \frac{4\mu}{\ell^2}}\right)$

· requiring no conical sing. as $r \to r_+$ fixes $\beta = \frac{2\pi \ell^2 r_+}{2r_+^2 + \ell^2}$

(only consider smooth saddle pts)

fixes rel. b/w β & μ.

· note that $\beta < \beta_{max} = \frac{\pi \ell}{\sqrt{2}}$ (for $r_+ = \frac{\ell}{\sqrt{2}}$) $\quad \frac{2ab}{a^2+b^2} \leq 1$

$\Rightarrow \exists$ a minimum temperature $T_{min} = \frac{\sqrt{2}}{\pi \ell}$ below which $\nexists$ black hole.

· $\forall\, T > T_{min}$, $\exists$ two black hole sols

$$r_+ = \frac{\pi \ell^2}{2\beta}\left[1 \underset{\text{small}}{\overset{\text{large}}{\pm}} \sqrt{1 - \frac{2\beta^2}{\pi^2 \ell^2}}\right] \qquad (r_+)_{small} < \ell$$

· the small b.h. can be $\ll \ell_{AdS}$ & will turn out to be thermodyn. unstab

$\approx$ Schw. in flat sp, they Hawking evap

· the large b.h is thermodyn stable ($C > 0$) & can thus be in thermal equil w/ thermal rad that bounces off AdS asympt.

· the free energy of the sols is def. via $Z = e^{-\beta F}$ & is computed by the on-shell Einstein action

$$S_E = -\frac{1}{16\pi G_N}\int d^5x\sqrt{g}\left(R + \frac{12}{\ell^2}\right) + \frac{1}{8\pi G_N}\int d^4x\sqrt{\gamma}\left(K + \frac{3}{\ell} + \frac{\ell R[\gamma]}{4}\right)$$

E.H. w/ c.c. GH counterterms

cutoff @ $z=\varepsilon$ & renormalize

$R \sim -\frac{20}{\ell^2}$ $\qquad I_{EH} = +\frac{1}{2\pi G_N \ell^2}\int_{r<\frac{1}{\varepsilon}}\sqrt{g}\, d^5x = +\frac{\beta\, \mathrm{vol}(S^3)}{2\pi G_N \ell^2}\int_{r_+}^{1/\varepsilon} dr\, r^3$

$\mathrm{vol}\, S^3 = \int \sin^2\psi \sin\theta\, d\theta\, d\varphi\, d\psi = 2\pi \cdot 2 \cdot \frac{\pi}{2} = 2\pi^2$

$$= \frac{\pi\beta}{4G_N\ell^2}\left(\frac{1}{\varepsilon^4} - r_+^4\right)$$

$$S_E = \frac{3\pi^2\ell^2\beta}{32\pi G_N} + \frac{2\pi^2\beta}{8\pi G_N \ell^2}\left(m\ell^2 - r_+^4\right)$$

($m = \frac{\mu}{2}$)

$$= \frac{\pi\beta}{4G_N\ell^2}\left(\frac{\mu}{2}\ell^2 - r_+^4 + \frac{3}{8}\ell^4\right) \qquad \ell^2\mu = \frac{r_+^4}{\ell^2} + r_+^2\ell^2$$

$$\boxed{= \frac{\pi\beta}{8G_N\ell^2}\left(-r_+^4 + \frac{3\ell^4}{4} + r_+^2\ell^2\right)}$$

$$S_E^{\text{small b.h.}} - S_E^{\text{large b.h.}} = \frac{\pi\beta}{8G_N\ell^2}\Big(\underbrace{(r_+^2)_{\text{small}} - (r_+^2)_{\text{large}}}_{<0}\Big)\Big(\ell^2 - \underbrace{(r_+^2)_{\text{sm}} - (r_+^2)_{\text{large}}}\Big)$$

$$\ell^2 - \left(\frac{\pi\ell^2}{2\beta}\right)^2\left(2 - \frac{2\beta^2}{\pi^2\ell^2}\right) \times 2 \qquad > 0$$

Factors π (Redo!)

$$= \frac{\pi\beta}{8G_N\ell^2}\cdot 2\ell^2\underbrace{\left(1 - \left(\frac{\pi\ell}{\beta\sqrt{2}}\right)^2\right)}_{\leq 0}\underbrace{\left((r_+^2)_{\text{sm}} - (r_+^2)_{\text{large}}\right)}_{<0}$$

⇒ the small b.h. always has **larger free eng than the large one** (never dominates over it in the canonical ensemble)

· $\langle E\rangle = -\partial_\beta \ln Z = \frac{3\pi}{8G_N}\left(\mu + \frac{\ell^2}{4}\right)$ Casimir eng.

· can check heat capacity

- $\exists$ another sol^n that satisfies the asympt AdS_5 $(S^3 \times S^1_\beta)$ bnd. cond, which is thermal AdS

no longer contractible

$$ds^2 = \left(1+\frac{r^2}{\ell^2}\right)dt_E^2 + \frac{dr^2}{1+r^2/\ell^2} + r^2 d\Omega_3^2 \qquad t_E \sim t_E + \beta$$

- β is now a free param (metric @ $r=0$ regular $\forall \beta$)

- note that while the metric is same as thermal AdS, the quantum state of pert. excit. on top of this geom. is $\neq$ from vacuum, & they do carry an $O(1)$ free energy whose backreaction on the geom is being neglected. (same as sd box $E \sim \ell^4 T^5$)

- since it is the sol'n w/ $\mu = 0$, its renormalized euclidean action is $S^{AdS}_{th} = \frac{\pi\beta}{8G_N\ell^2} \cdot \frac{3\ell^4}{4}$ This should however be treated as zero, as it's pure Casimir eg. contr. In any case

$$S^{large}_{bh} - S_{th.AdS} = \frac{\pi^2\beta}{8G_N\ell^2}\, r_+^2(\ell^2 - r_+^2)$$

$$\left(r_+^2\right)_{larg} = \left(\frac{\pi\ell^2}{2\beta}\right)^2 \left(2 - \frac{2\beta^2}{\pi^2\ell^2} + 2\sqrt{1-\frac{2\beta^2}{\pi^2\ell^2}} - \frac{4\beta^2}{\pi^2\ell^2}\right) > 0 \quad \text{for } \beta < \frac{2\pi\ell}{3}$$

$> \ell^2$

- thus, the black hole has smaller free energy for $\beta < \frac{2\pi\ell}{3}$ (high temperatures $T > \frac{3}{2\pi\ell}$), while thermal AdS is preferred for $T < \frac{3}{2\pi\ell}$. For $T < T_{min}$, the b.h. sol'n ceases to $\exists$.
This is called the Hawking-Page transition $\to$ exchange of dominance of sol's w/ $\neq$ topologies

- notice that, before the transition $S \approx 0$ (or an $O(G_N^0)$ #) coming from the matter fields

- after the transition, the entropy is $S_{BH} = \dfrac{\pi^2 r_t^3}{2G_N} \sim O(N^2)$

  - we go from a very small # states @ low temp to a very large one @ high temp → interpreted as confinement / deconfinement transition in the dual $\mathcal{N}=4$ SYM.

  - disappears if $S^3 \to \mathbb{R}^3$. This is b/c decomp. same as $\beta \to 0$ b/c only $\beta/\text{size}\,S^3$ is a param. Then $r_t = \dfrac{\pi \ell^2}{\beta}$ & the free energy $F = \dfrac{1}{\beta} S_E = \dfrac{-\pi}{8G_N \ell^2}\left(\dfrac{\pi \ell^2}{\beta}\right)^4 = -\dfrac{\pi^5 \ell^6}{8G_N} T^4$

$\text{vol}\,S^3 = \ell^3 \cdot 2\pi^2$ $\quad F = -\dfrac{\pi^3 \ell^3}{16 G_N} T^4 V_{S^3}$ as exp. from conformal invariance & F = extensive

$$\frac{1}{16\pi G_5} = \frac{\ell^5\, \text{vol}\, S^5\, \pi^3}{(2\pi)^7 \alpha'^4 g_s^2} \qquad \lambda = 4\pi g_s N = \frac{\ell^4}{\alpha'^2} \qquad \Big| \quad \frac{\ell^3}{16 G_5} = \frac{\pi^4 \ell^8}{2^7 \pi^7 \left(\frac{\ell^4}{4\pi N}\right)^2 8} = \frac{16\pi^2 N^2}{2^7 \pi^3}$$

$$\Rightarrow F = -\frac{\pi^2 N^2}{8} T^4 V \qquad \qquad \Rightarrow \frac{\ell^3}{4G_N} = \frac{N^2}{2\pi}$$

- this may be compared w/ a computation of the free eng in free SYM

$$F = -\frac{\pi^2}{6} N^2 T^4 V \qquad \text{don't agree}$$

identif w/ T_H

- this is not surprising, given that we're comparing strong & weak coupling & there is no reason F is protected (susy)

Black holes in AdS_3

- another set of AdS b.hs that have been very extensively studied are the so-called BTZ b.h. in AdS_3
- they are sol's of pure 3d gravity w/ $\Lambda < 0$ $\quad S = \frac{1}{16\pi G_3}\int d^3x \sqrt{g}\,(R + \frac{2}{\ell^2}$
- in 3d, gravity is not dyn. b/c $R_{\mu\nu\rho\sigma}$ = det by $R_{\mu\nu}$ ← fixed by Einstein's eqns. $\quad R_{\mu\nu} = -\frac{2}{\ell^2} g_{\mu\nu}$ ← check $\Rightarrow$
  $R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$ same as in empty AdS_3
- thus, all sols of pure 3d gravity w/ c.c. <0 must be locally AdS_3
- the BTZ b.h. corresp. to a quotient of global AdS_3 that has the causal str. of a b.h. It does carry non-trivial M, J. For J=0 (for simplicity)

$$ds^2 = -\left(\frac{r^2}{\ell^2} - 8GM\right)dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2} - 8GM} + r^2 d\varphi^2$$

$\varphi \sim \varphi + 2\pi$

- the Hawking temp is T_H
  - a single sol'n for fixed T → analogue of the large AdS b.h. (small absent)
  - thermodyn. stable
- the entropy is $S_{BH} = \frac{2\pi r_+}{4G_N} = \frac{\pi \ell}{2G_{N_3}}\sqrt{8GM} = 2\pi\sqrt{\frac{\ell^2 M}{2G}}$

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· it is interesting to note that, in this case (unlike that of AdS_5 black holes) it is possible to reproduce the entropy exactly (incl. the numerical coeff) from a microscopic computation.

· the reason is that in 2d, modular invariance allows us to compute the CFT_2 finite temp. entropy even at strong coupling

path int $\underset{R}{\boxed{}}\,\beta \;=\; Z_\beta(R) = Z_R(\beta)$

Using conf. invar $Z = Z(\beta/R)$ $\quad Z_\beta(R) = Z_{\frac{R^2}{\beta}}(R)$

$$Z(\beta) = Z\left(\frac{4\pi^2}{\beta}\right) \quad \text{for } R = 2\pi$$

relates low & high temp. $\underset{\substack{\to 0 \\ \text{high temp}}}{Z(\beta)} = \underset{\text{low}}{Z\left(\frac{4\pi^2}{\beta}\right)} \approx e^{-\frac{4\pi^2}{\beta} E_{vac}}$ $\quad E_{vac} = -\frac{c}{12}$

· end result

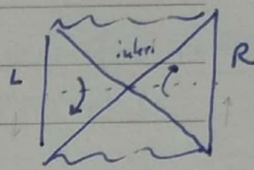
$$S_{cardy} = 2\pi\sqrt{\frac{c}{3} E}$$

: using our previous holographic result that $c = \frac{3\ell}{2G_3}$, we find precisely that $S_{BH} = 2\pi\sqrt{\frac{3\ell}{2G} \cdot \frac{M\ell}{3}} = 2\pi\sqrt{\frac{c}{3} M\ell}$

ℓ radius circle.

· thus, b.h. entropy counts high-energy excit. in the dual CFT

Eternal black holes & entanglement

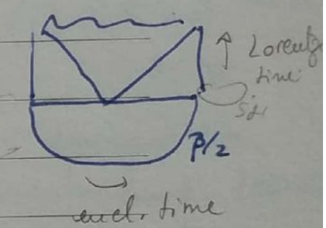


- consider the eternal (large) AdS-Schw. b.h.
  - maximal analytic ext. of Schw-AdS spt. (≠ b.h. collapse)

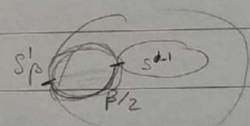
$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2 + \text{ext.}$$

- 2 bnd → 2 copies of the dual CFT, state in $\mathcal{H}_L \otimes \mathcal{H}_R$ (identical CFTs)
  (as det. by bnd. cond. near the 2 AdS bnd)

- cutting the Lorentzian geom. @ $t=0$ (time refl.-symm.), we can prepare the state of the quantum fields on the $t=0$ slice via a euclidean path. int, obtained by cutting the eucl. b.h. geom. in half.



This is the Hartle-Hawking constr. of the wavef.



- the question is: what is the corresp. state in the dual $CFT_L \times CFT_R$?

- via AdS/CFT $\quad Z_{bulk}[\ \beta/2\ ] = Z_{CFT}[\ \ ]$
  half-bagel

$$= Z_{CFT}[I_{\beta/2} \times S^{d-1}]$$

the claim is that the CFT^2 path int produces the thermofield double st.

$$|TFD\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta E_n}{2}} |n\rangle_L |n\rangle_R$$

eigenst. of single CFT

w/ prop. that tracing over L/R d.o.f ⇒ perf. thermal ρ_R / ρ_L

$Z_{CFT}[I_{\beta/2} \times S^{d-1}]$

φ_1 O——O φ_2 ($\beta/2$) $= \langle \varphi_1 | e^{-\frac{\beta H}{2}} | \varphi_2 \rangle = \sum_n \langle \varphi_1 | n \rangle \langle n | \varphi_2 \rangle e^{-\frac{\beta E_n}{2}} = \sum_n \langle \varphi_1 | \langle \varphi_2^* | e^{-\frac{\beta E_n}{2}} |n_L\rangle |n_R\rangle$

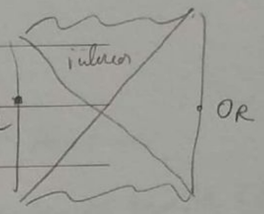
$\langle \varphi_2^* | n^* \rangle$

* → reverses direction of time

$\langle \psi | \Theta | \ \rangle\ |\psi^*\rangle$

· thus, we find the rather remarkable fact that, even though the 2 CFTs are entirely non-interacting, ∃ a geometric connection b/w them, due to the entanglement in the TFD state.

· this can be probed by e.g. inserting ops.



$$\langle \psi_{TFD} | O_L O_R | \psi_{TFD} \rangle$$

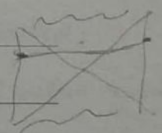
– in CFT, $\neq 0$ due to entanglement $\sim \sum_{n,n'} e^{-\beta \frac{E_n + E_{n'}}{2}} \langle n | O_L | n' \rangle \langle n | O_R | n' \rangle$

– in gravity, $\neq 0$ b/c ∃ a geom conn $\langle O_L O_R \rangle \approx e^{-\Delta l_{LR}}$ Δ large.

· this geom. conn. is through the b.h. interior + emergent due to entanglement. (ER=EPR)

· one may use this setup to devise a simpler version of the info paradox.

– fix $O_L(0)$ & take $O_R(t)$, t very large. Gravity suggests that $\langle O_L O_R \rangle \sim e^{-\frac{\Delta t}{\beta}}$ indefinitely (& we can wait b/c b.h. is eternal)



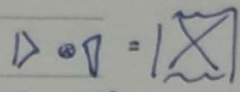
· however, CFT arg. indicate it becomes $O(e^{-S})$ @ large times

$$\langle O_L O_R^{(t)} \rangle = \frac{1}{Z^2} \underbrace{\sum_{n,n'}}_{\sim e^{2S}} \overbrace{\langle n | O_L | n' \rangle \langle n | O_R | n' \rangle}^{O(1) \text{ for } n'=n} \underbrace{e^{-i(E_n - E_{n'})t}}_{\text{averages to zero for } n \neq n' \; e^{2S} \to e^{S}} e^{-\beta \frac{(E_n + E_{n'})}{2}} \sim e^{-S}$$

· sharp version of the paradox → can in principle be resolved via non-pert. corr. ($\sim e^{-S}$)

(Malda suggested incl. eucl AdS saddle, Barbon-Rabinovici argued not enough, for JT SSS argued plateau comes from "D-branes")

Emergent geometry & entanglement

- the literal int. of the TFD constr is that $\sum_n e^{-\beta E_n/2}$  $|E_n\rangle$ $|E_n\rangle$

  $\sum$ disconn. geom $\Rightarrow$ connected

  suggesting entanglement $\overset{\text{rel. to}}{\leftrightsquigarrow}$ connectedness of spt.

- this can be (semi) sort-of quatitatively motivated by the fact that in this TFD constr, the b.h. entropy $\leftrightarrow$ entanglement entropy b/w the 2 CFTs

- taking CFT_2 for concreteness, we have $S_{\text{bh}} \sim c\,T$, which decreases as T is decreased

  van Raamsdonk
  1005.2025
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- also, the mutual info b/w the 2 CFTs $I(A,B) = S(A) + S(B) - S(A\cup B)$

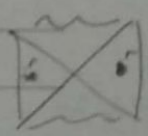
  if I decreases, then corr. b/w the 2 subsystems must decrease, since $I(A,B)$ provides an upper bnd. for them

$$I(A,B) \geq \frac{(\langle O_A O_B\rangle - \langle O_A\rangle\langle O_B\rangle)^2}{2|O_A|^2\,|O_B|^2}$$

  these corr. can be seen to become smaller as the temp. of the b.h. is reduced. In Kruskal coord, the BTZ metric reads

$$ds^2 = -\frac{4\,du\,dv}{(1+uv)^2} + 4\pi^2T^2\frac{(1-uv)^2}{(1+uv)^2}\,d\varphi^2 \qquad \text{letting } \begin{matrix} u = t+x \\ v = t-x \end{matrix}$$

  apparently computing lengths b/w. 2 sides when (@ $t=0$) φ circumf is $2\pi r$, we find



$$L \sim 2\ln\frac{\sqrt{\frac{r}{2\pi T}+1} + \sqrt{\frac{r}{2\pi T}-1}}{\sqrt{\frac{r}{2\pi T}+1} - \sqrt{\frac{r}{2\pi T}-1}} \to \infty \text{ as } T\to 0$$

  so corr $\to 0$

- so, as T decreases, entanglmt & corr. b/w two sides are reduced.

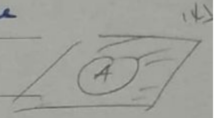
  supporting the idea that cls connectivity arises by (Large) entanglement.

- relevant for more gen. quest. about how spt. emerges from / is encoded in geom.

Holographic entanglement entropy

- you saw how, in the AdS/CFT context, the Bekenstein-Hawking formula provides a geometric interpretation for the thermal CFT entropy
  coarse grained (max ρ subject to $\mathrm{Tr}\,\rho A = \rho_{\dots}$)

- you also learned that, in a local QFT, it is possible to define an entanglement entropy assoc. w/ a spatial subregion A



via
$$S(A) = -\mathrm{Tr}\,\rho_A \ln \rho_A \qquad \rho_A = \mathrm{Tr}_{A^c} |\psi\rangle\langle\psi|$$
← global state system.

- fine-grained (much more detailed measure of the state $|\psi\rangle$)

shows that
- even in QFT vacuum, $\exists$ lots of entanglement, most coming from correlations just across the entangling surface

$$S_A \sim \# \,\underbrace{\mathrm{area}(\partial A)}_{L_A^{d-2}} + \# L^{d-4} \quad \text{\& log terms in even } d.$$
($\#$ theory-dep.; univ. div, $\forall$ state)

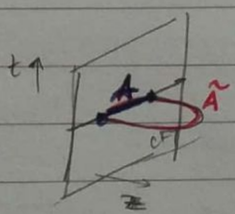
. In highly excited st, exp. $S_A \sim \mathrm{vol}(A)$.

(in CFT_d w/ no other scales $\# \left(\frac{\ell_A}{\varepsilon_{uv}}\right)^{d-2} + \# \left(\frac{\ell_A}{\varepsilon_{uv}}\right)^{d-4} + \dots$

. coeff. log (even d) $\propto$ central ch, (a in 4d) & of 1° (odd d) ~ sphere part. f. (monotonous along RG flows)

- $\exists$ a holographic int of S_A?

  - consider hologr. CFT_d (large N, large gap) in a state $|\psi\rangle \leftrightarrow$ smooth classical geom*
    & consider spatial subregion A



RT proposal

$$S_A = \frac{\mathrm{Area}(\tilde{A})}{4G_N}$$

. area cod 2 bulk surface of min area ~~that has~~ same bnd. as A.

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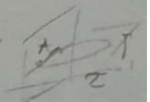
* Einstein gravity dual

- in addition, the bulk surface $\tilde{A}$ is homologous to A (can be cont. def into each other in the bulk) (! Check)

- $\tilde{A}$ extremizes area functional. If several sol'ns, take one w/ smallest area

(∃ generaliz (HRT) for time-dep backgnds, whereby one 1st minimizes area on a bulk spatial slice $\Sigma \supset A$, but then maximizes it over all possible Σ)



- as in the CFT calc, this quantity is divergent due to the ∞ volume of AdS → structure div. matches CFT (standard UV/IR connection)

  ill def / regulated by cutoff @ $z=\varepsilon$

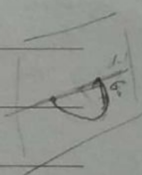
- could also consider cutoff-indep quantities, ∋ mutual info, $S_A(4S) - S_A(10S)$, $\frac{\partial S_L}{\partial L}$ etc

Example : ball-shaped region in CFT vacuum.

- dual geom Poincaré AdS $\quad ds^2 = \frac{\ell^2}{z^2}\left(dz^2 - dt^2 + dx^i dx^i\right) = G_{MN}\, dx^M dx^N \qquad x^i \in (1, \dots, d-1)$

surface lies @ $t=0$.

$$\text{Area} = \int d^{d-1}\sigma \sqrt{\det g_{ab}} \qquad g_{ab} = G_{MN} \frac{\partial x^M}{\partial x^a}\frac{\partial x^N}{\partial x^b}$$



- param by x^i & $z(x^i)$

$$g_{ij} = \frac{\ell^2}{z^2(x_i)}\left(\delta_{ij} + \frac{\partial z}{\partial x^i}\frac{\partial z}{\partial x^j}\right) \qquad \begin{pmatrix} 1+\partial_x z^2 & \partial_x z\, \partial_y z \\ \partial_x z\, \partial_y z & 1+\partial_y z^2 \end{pmatrix} = 1+\partial$$

$$\sqrt{\det g_{ij}} = \frac{\ell^{d-1}}{z^{d-1}}\sqrt{1 + \frac{\partial z}{\partial x^i}\frac{\partial z}{\partial x^i}}$$

ball → spherical symm $dx^i dx^i = dr^2 + r^2 d\Omega^2_{d-2}$, $z = z(r)$ $\quad \mathrm{vol}\, S_{d-2} \int \frac{r^{d-2}\, dr}{z(r)^{d-1}}\sqrt{1+(\partial_r z)^2}$

minimize $\frac{r^{d-2}}{z^{d-1}}\sqrt{1+(\partial_r z)^2}$ w/ soln $z^2 + r^2 = \text{const} = R^2$ ← radius ball on the bnd.

- calculate area for e.g. $d=2$ $\quad A = \ell \int \frac{dr}{z}\sqrt{1+(\partial_r z)^2} = \ell \cdot R \int_0^{R-\eta} \frac{dr}{z^2} \; (\times 2)$ $\qquad \left(\frac{r^2}{z^2}\right)$

where η is such that $\varepsilon^2 + (R-\eta)^2 = R^2$

$\varepsilon^2 - 2R\eta = 0$

thus $S_R = \frac{\ell R}{4G_3} \cdot 2 \, \frac{1}{R} \, \text{arcth} \frac{r_c}{R} \Big|_0^{R - \frac{\varepsilon^2}{2R}} = \frac{\ell}{2G_3} \, \underbrace{\text{arcth}\left(1 - \frac{\varepsilon^2}{2R^2}\right)}$

$\approx -\frac{1}{2} \ln\left(\frac{\varepsilon^2}{4R^2}\right)_{L^2}$

$$\boxed{\underset{\substack{\to L \\ \text{total length of} \\ \text{CFT interval}}}{S^{CFT_2}_L} = + \frac{c}{3} \ln \frac{L}{\varepsilon}}$$ R ∞ length

exactly the same as CFT_2 vacuum answer.

omit or look up

- special to single interval (univ. answer depending on just anomaly coeff)

multiple interval cases depend on CFT details & only agrees w/ gravity for large c)

– thus, @ least for $d = 2$, we have checked RT yields right answer (single int)

– also check thermal 2d, higher d Balls + deformation

Holographic entanglement @ finite temp.

· consider $t = 0$ slice BTZ metric

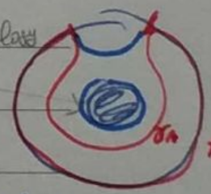
$$dr^2 = \frac{dr^2}{r^2 - 8GM} + r^2 d\varphi^2 \qquad T_H = \frac{\sqrt{8GM}}{2\pi \ell}$$

$\in (0, R)$

· $R \ll 2\pi$, S_A computed by length shortest (! ∞, reg!) geod in BTZ geom

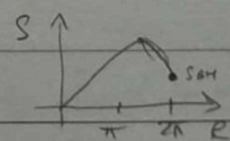
$$S_A = \frac{c}{3} \ln\left[\frac{1}{\pi T \varepsilon_{uv}} \text{sh}(\pi R T)\right]$$

due to homology cond



· for $R \sim 2\pi$, this is not necessarily the shortest geod. (also blue one)

length blue curve $S_A^{blue} = \frac{c}{3} \ln\left[\frac{1}{\pi T \varepsilon_{uv}} \text{sh}(\pi(2\pi - R)T)\right] + \underbrace{\frac{2\pi^2}{3} c T}_{\text{b.h. entropy}}$



· must choose $S_A = \min\left[S_A^{red}, S_A^{blue}\right]$

exchange domin @ some $R > \pi$

notice $S_A \neq S_A^c$ b/c. of b.h. contr.

optional

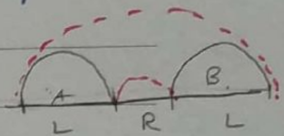
Phase transition in the mutual info

- we saw the Bekenstein-Hawking entropy can undergo phase transitions as a $f(T)$. Similar behaviours can be found in the entanglement entropy

(significance?)

- better work w/ the mutual info $I(A:B) = S(A) + S(B) - S(A \cup B)$ which has the advantage of being finite. Setup:

  · 2 regions size L separated by R



$$S(A \cup B)^{\text{alt}} = \frac{c}{3} \ln \frac{R+2L}{\varepsilon} + \frac{c}{3} \ln \frac{R}{\varepsilon} < \frac{c}{3} \ln \frac{L}{\varepsilon} \times 2 \quad \text{for}$$

$$R(R+2L) < L^2, \text{ or } \quad R < L(-1+\sqrt{2})$$

the mutual info is $I(A:B) = \frac{c}{3} \ln \frac{L^2}{R(R+2L)}$ for $R < L(\sqrt{2}-1)$

0 for $R > L(\sqrt{2}-1)$

· interesting first order phase transition as R is increased.

- applications of this?

Implications of the holography ent. entropy formula

$S_A = \frac{\text{Area}(\tilde{A})}{4G_N}$, $\forall \tilde{A}$ → (over) det. in principle the spt. geom. in terms of the entanglmt str. of the CFT

↦ highly overconstrained pb, since ∃ many more subregion shapes than bulk geom

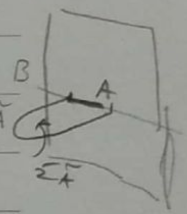
↦ w/ few exceptions (2dCFT) currently hard to underst. why, @ large N & strong coupling CFT entang behaves this way.

• while RT simply conjectured their formula, note ∃ a bulk derivation for it, i.e. upon assuming that the CFT is descr. by classical Einst. grav. in an a AdS spt.

• the deriv (Malda - Lewkowycz) computes the $Z_{CFT}[B_q]$ using the hologr. descr. that rel. it to $Z_{bulk}[B_q]$ which is extremized by a smooth bulk manif. that ends on B_q @ the bnd.

bnd. replica manif to evaluate Renyi

$$S_A = \underbrace{\frac{\mathcal{A}(\tilde{A})}{4G_N}}_{O(N^2)} + \underbrace{S_{bulk}(\Sigma_{\tilde{A}})}_{O(1)}$$



• additionally, they can gen their prescr. to include quantum corr / $\frac{1}{N}$ corr. that turns out to equal the ent. entropy of bulk fields across the extremal surf.

↦ very similar in spirit w/ S_{gen}

↦ note that in the above, we first found the min. area surface, & then computed $S_{bulk}(\Sigma_{\tilde{A}})$ across this surface, however, it has been argued one should instead minimize the full S_{gen} → entropy assoc w/ a quantum extremal surface (allows $\mathcal{A}_{min} = \langle \underbrace{\hat{\mathcal{A}}}_{\text{area op}} \rangle$ to resp. to quantum fluct. of the geom) (relevant @ subleading order)

↦ notice this is just a classical surface but whose location is det also by the state of the quantum fields

• this formula will be essential in recovering a unitary Page curve for b.h. evaporation, as Jildou will show.