Petnica Summer School

Problem Sets

Exercise 1

Start with the FLRW metric and write it in conformal time. Why is it useful? (set $K=0$). Define the conformal Hubble parameter $\mathcal{H} = \frac{a'}{a}$; how is it related to H? How does $\mathcal{H}^{-1}$ evolve in RD/MD? And DE? Write both Friedmann eq.s in terms of $\mathcal{H}$ and τ.

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right)$$

$$ds^2 = a^2(\tau)\left(-d\tau^2 + \frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right)$$

$K=0 \longrightarrow$ conformally Minkowski

$$\mathcal{H} = \frac{a'}{a} = \frac{1}{a}\frac{da}{d\tau} = \frac{1}{a} a \frac{da}{dt} = aH$$

$$\mathcal{H}^{-1} = \frac{1}{aH} = \frac{1}{\dot{a}}$$

MD: $a \propto t^{2/3} \longrightarrow \dot{a} \propto t^{-1/3}$

RD: $a \propto t^{1/2} \longrightarrow \dot{a} \propto t^{-1/2}$

$$\mathcal{H}^{-1} \propto \begin{cases} t^{1/3} & \text{RD} \\ t^{1/2} & \text{MD} \end{cases} \longrightarrow \text{power law increasing}$$

DE: $a \propto e^{Ht} \longrightarrow \dot{a} \propto e^{Ht}$

$\mathcal{H}^{-1} \propto e^{-Ht} \longrightarrow$ it decreases with time!

$$\begin{cases} H^2 = \dfrac{8\pi}{3}\rho - \dfrac{K}{a^2} \\ \dot{H} + H^2 = \dfrac{4\pi}{3}(\rho + 3p) \end{cases}$$

$$\begin{cases} \mathcal{H} = aH \\ H^2 = \dfrac{\mathcal{H}^2}{a^2} \\ d/dt = \dfrac{1}{a}\, d/d\tau \end{cases}$$

$$\begin{cases} \mathcal{H}^2 = \dfrac{8\pi}{3}a^2\rho - K \\ \dfrac{1}{a}\dfrac{d}{d\tau}\left(\dfrac{\mathcal{H}}{a}\right) + \dfrac{\mathcal{H}^2}{a^2} = -\dfrac{4\pi}{3}(\rho + 3p) \end{cases}$$

$$\frac{1}{a}\left(\frac{\mathcal{H}'}{a}\right) - \cancel{\frac{1}{a}\frac{a'}{a^2}\mathcal{H}} + \cancel{\frac{\mathcal{H}^2}{a^2}} = -\frac{4\pi}{3}(\rho + 3p)$$

$$\mathcal{H}' = -\frac{4\pi}{3}a^2(\rho + 3p)$$

$$\begin{cases} \mathcal{H}^2 = \dfrac{8\pi}{3}a^2\rho - K \\ \mathcal{H}' = -\dfrac{4\pi}{3}a^2(\rho + 3p) \end{cases}$$

EXERCISE 2

Compute the comoving distance at which we see the photons as steady.
Is it possible we see an object moving faster than light? If yes, explain why.

$$r_{ph} = ar$$

$$v_o = \frac{d}{dt}(ar) = H r_{ph} + v_{ph} \longrightarrow v_{ph} = a\frac{dr}{dt} = c = 1$$

$$v_o = 0 \longrightarrow H r_{ph} + v_{ph} = 0$$

$$Har + 1 = 0 \longrightarrow r = -\frac{1}{aH}$$

Exercise 3

Show using the 2nd F. eq. and $p = w\rho$ that $\ddot{a} > 0 \iff w < -\frac{1}{3}$.

$$p = w\rho$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p) = -\frac{4\pi}{3}\rho(1+3w)$$

$$\ddot{a} = -\frac{4\pi}{3}\rho a(1+3w)$$

$$\ddot{a} > 0 \longrightarrow 1+3w < 0 \longrightarrow w < -1/3$$

Exercise 4

Find the integral relation between t and τ. Rewrite the definition of comoving particle horizon d_p in terms of $\ln a$ and the Hubble radius H. Then write the comoving Hubble radius $1/aH$ in terms of a only (hint: use 1st F. eq. and equation of state; set $K=0$). Integrate the expression you found to get a result for d_p. What is the dominant contribution in the ΛCDM model (RD + MD). What happens if we assume there is a period of $\ddot{a} > 0$ in the Early Universe?

$$dt = a\, d\tau \longrightarrow \tau - \tau_0 = \int_{t_0}^{t} \frac{dt'}{a(t')}$$

Comoving particle horizon: $\quad r_{ph} = a\, r_{comoving}$

$$d_p^{com}(\tau) = \int_{t_0}^{t} \frac{dt'}{a(t')} = \tau - \tau_0$$

$$= \int_{t_0}^{t} \frac{1}{a(t')} \frac{dt'}{da} da = \int_{a_i}^{a} \frac{da}{a\dot{a}} =$$

$$= \int_{\ln a_i}^{\ln a} \frac{1}{\dot{a}} d\ln a = \int_{\ln a_i}^{\ln a} \frac{1}{aH} d\ln a$$

$$\begin{cases} H^2 = \frac{8\pi}{3}\rho \\ \dot{\rho} + 3H(\rho + p) = 0 \end{cases} , \quad p = w\rho$$

$$\longrightarrow \frac{d\rho}{dt} = -3H\rho(1+w) = -3(1+w)\frac{\dot a}{a}$$

$$\frac{d\rho}{\rho} = -3(1+w)\,H(t)\,dt = -3(1+w)\frac{da}{a}$$

$$\ln(\rho-\rho_i) = -3(1+w)\ln(a-a_i)$$

$$\rho = \rho_i\left(\frac{a}{a_i}\right)^{-3(1+w)}$$

$$\longrightarrow H = \sqrt{\frac{8\pi\rho}{3}} = \sqrt{\frac{8\pi\rho_0}{3}}\left(\frac{a}{a_0}\right)^{-3/2(1+w)}$$

$$\frac{1}{aH} = \underbrace{\boxed{\sqrt{\frac{3}{8\pi\rho_i}}\,a_i^{3/2(1+w)}}}_{c}\,a^{1/2(1+3w)}$$

$$\longrightarrow d_p(\tau) = \int_{\ln a_i}^{\ln a} c\,a^{1/2(1+3w)}\,d\ln a =$$

$$= \int_{\ln a_i}^{\ln a} c\,e^{\frac{1}{2}(1+3w)\ln a}\,d\ln a =$$

$$= \frac{2c}{1+3w}\left[e^{1/2(1+3w)\ln a} - e^{1/2(1+3w)\ln a_i}\right] =$$

$$= \frac{2c}{1+3w}\left[a^{1/2(1+3w)} - a_i^{1/2(1+3w)}\right]$$

In standard BB scenario $a \gg a_i \longrightarrow d_p(\tau) \approx \frac{2c}{1+3w}\,a^{1/2(1+3w)}$

Late time contribution dominates. Set $\tau_i = 0$ for $a_i = 0 \longrightarrow d_p = \tau$.

Notice that in this case $d_p(\tau) \propto 1/aH$; $w = 1/3 \longrightarrow d_p = d_H$

$w = 0 \longrightarrow d_p = 2d_H$

If $\ddot{a} > 0 \longrightarrow w < -\frac{1}{3} \longrightarrow (1+3w) < 0$

$\longrightarrow \frac{d}{dt}\left(\frac{1}{aH}\right) < 0$, Hubble radius shrinks

$\longrightarrow$ major contribution from early times

Indeed $\eta_i = \frac{2c}{\underbrace{(1+3w)}_{<0}} a_i^{\frac{1}{2}\overbrace{(1+3w)}^{<0}} \xrightarrow[1+3w<0]{a_i \to 0} -\infty$

If in the standard scenario $\eta_i = 0$, if initially $\ddot{a} > 0$ we push to $\eta_i = -\infty$ the starting point i ($=$ BB).

Exercise 5

Derive the redshift at equality knowing Ω_m° and Ω_r°.
Derive $\Omega_k(t)$ knowing Ω_k° and Ω_m°.

$$\Omega_m = \Omega_r \longrightarrow a_{eq} = \frac{\Omega_r^\circ}{\Omega_m^\circ} = \frac{1}{1+z_{eq}} \longrightarrow z_{eq} = \frac{\Omega_m^\circ}{\Omega_r^\circ} - 1$$

$$\Omega_m^\circ \sim 0.3 \,/\, \Omega_r^\circ \sim 8.4 \times 10^{-5} \longrightarrow z_{eq} \approx 3570$$

$$\frac{\Omega_k(t)}{\Omega_k^\circ} = \frac{(a_0 H_0)^2}{(aH)^2}$$

$$\text{MD + RD}: \quad H^2 = H_0^2 \left[\Omega_m^\circ \left(\frac{a_0}{a}\right)^3 + \Omega_r^\circ \left(\frac{a_0}{a}\right)^4 \right]$$

At equality: $\Omega_r = \Omega_m$

$$\longrightarrow \Omega_m^\circ = \Omega_r^\circ \frac{a_0}{a_{eq}} \longrightarrow \Omega_r^\circ = a_{eq} \Omega_m^\circ$$

$$\longrightarrow H^2 = H_0^2 \Omega_m^\circ \left[\left(\frac{a_0}{a}\right)^3 + a_{eq} \left(\frac{a_0}{a}\right)^4 \right] =$$

$$= H_0^2 \Omega_m^\circ \left(\frac{a_0}{a}\right)^4 \left[\frac{a}{a_0} + a_{eq} \right] =$$

$$= H_0^2 \Omega_m^\circ a^{-4} (a + a_{eq})$$

$$\longrightarrow \Omega_k(t) = \Omega_k^\circ \frac{(\cancel{a_0} \cancel{H_0})^2}{a^2 \cancel{H_0^2} \Omega_m^\circ a^{-4} (a + a_{eq})} =$$

$$= \frac{\Omega_k^\circ}{\Omega_m^\circ} \frac{a^2}{(a + a_{eq})}$$

Exercise 6

2 component Universe: find a solution for the scale factor in the presence of both matter and radiation.
(hint: work in conformal time)

$$\begin{cases} \mathcal{H}^2 = \frac{8\pi}{3} a^2 \rho_t & \longrightarrow \quad a'^2 = \frac{8\pi}{3} \rho_t a^4 \\ \mathcal{H}' = -\frac{4\pi}{3} a^2 (\rho_t + 3p_t) & \longrightarrow \quad \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = -\frac{4\pi}{3} (\rho_t + 3p_t) a^2 \end{cases}$$

$$\longrightarrow \frac{a''}{a} = -\frac{4\pi}{3} (\rho_t + 3p_t) a^2 + \frac{8\pi}{3} \rho_t a^2 = \frac{4\pi}{3} (\rho_t - 3p_t) a^2$$

$$a'' = \frac{4\pi}{3} (\rho_t - 3p_t) a^3$$

$$\rho_t = \rho_r + \rho_m$$

At equilibrium: $\rho_r = \rho_m = \frac{\rho_t^{eq}}{2}$

$$\rho_r = \rho_r^{eq} \left(\frac{a_{eq}}{a}\right)^4 = \frac{\rho_{eq}}{2} \left(\frac{a_{eq}}{a}\right)^4$$

$$\rho_m = \rho_m^{eq} \left(\frac{a_{eq}}{a}\right)^3 = \frac{\rho_{eq}}{2} \left(\frac{a_{eq}}{a}\right)^3$$

$$\longrightarrow \rho_t = \frac{\rho_{eq}}{2} \left[\left(\frac{a_{eq}}{a}\right)^3 + \left(\frac{a_{eq}}{a}\right)^4 \right]$$

$$\rho_r - 3p_r = 0 \qquad \left(p = \frac{1}{3}\rho \right)$$

$$p_m = 0$$

$$\longrightarrow \quad a'' = \frac{4\pi}{3} \rho_m a^3$$

$$\rho_m = \rho_m^{eq} \left(\frac{a_{eq}}{a} \right)^3 \longrightarrow \rho_m a^3 = \rho_m^{eq} a_{eq}^3 = \frac{1}{2} \rho_{eq} a_{eq}^3 = \text{const}$$

$$\longrightarrow \quad a'' = \frac{2\pi}{3} \rho_{eq} a_{eq}^3$$

$$\longrightarrow \quad a(\tau) = \frac{\pi}{3} \rho_{eq} a_{eq}^3 \tau^2 + c_1 \tau + c_2$$

$$a(\tau = 0) = 0 \longrightarrow c_2 = 0$$

Use 1st F. eq to determine c_1.

$$a(\tau) = \frac{\pi}{3} \rho_{eq} a_{eq}^3 \tau^2 + c_1 \tau$$

$$a'(\tau)^2 = \left[\frac{2\pi}{3} \rho_{eq} a_{eq}^3 \tau + c_1 \right]^2$$

1st F eq: $$a' = \left\{ \frac{8\pi}{3} a^4 \frac{\rho_{eq}}{2} \left[\left(\frac{a_{eq}}{a} \right)^3 + \left(\frac{a_{eq}}{a} \right)^4 \right] \right\}^{1/2} =$$

$$= \left\{ \frac{4\pi}{3} \rho_{eq} a_{eq}^3 \left(a + a_{eq} \right) \right\}^{1/2}$$

$$\longrightarrow (a')^2 = \frac{4\pi}{3} \rho_{eq} a_{eq}^3 \left[\frac{2\pi}{3} \rho_{eq} a_{eq}^3 \tau^2 + c_1 \tau + a_{eq} \right]$$

$$\longrightarrow c_1 = \left(\frac{4\pi}{3} \rho_{eq} a_{eq}^4 \right)^{1/2}$$

$$\longrightarrow \quad a(\tau) = \underbrace{\frac{\pi}{3}\rho_{eq}\, a_{eq}^3}_{a_{eq}\left(\pi/3\, \rho_{eq}\, a_{eq}^2\right)} \tau^2 + \underbrace{\left(\frac{4\pi}{3}\rho_{eq}\, a_{eq}^4\right)^{1/2}}_{2a_{eq}\left(\frac{\pi}{3}\rho_{eq}\, a_{eq}^2\right)^{1/2}} \tau =$$

$$= a_{eq}\left[\left(\frac{\tau}{\tau_*}\right)^2 + 2\left(\frac{\tau}{\tau_*}\right)\right]$$

$$\tau_* = \left(\frac{\pi}{3}\rho_{eq}\, a_{eq}^2\right)^{-1/2}$$

Exercise 7

From statistical mechanics considerations, we can write the energy density and the pressure of particles as:

$$\begin{cases} \rho_i = g_i \int \frac{d^3k}{(2\pi)^3} E f_i(E) \\ p_i = g_i \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{3E} f_i(E) \end{cases} \qquad E(\bar{k}) = \sqrt{|\bar{k}|^2 + m_i^2}$$

it quantifies the number of particles in a volume element of phase space $f(\bar{x}, \bar{p}, t)$

$$n_i = g_i \int \frac{d^3k}{(2\pi)^3} f_i(E) \qquad \text{(number density)}$$

where g_i are the degrees of freedom of particle i and f_i the phase-space distribution function. $f_i = f_i(E)$ in a homogeneous and isotropic Universe.

1) Show that for ultra-relativistic particles $(k \gg m)$ $p_i = \frac{1}{3} \rho_i$
2) Derive from the equation of state how ρ scales with the scale factor for both radiation (ultra-relativistic) and matter (non-relativistic)
3) Suppose the Universe is at thermal equilibrium, then

$$f_i = f_i^{eq} \equiv \frac{1}{\exp\left((E-\mu_i)/T\right) \pm 1}$$

$+$ fermions
$-$ bosons

with T a common temperature. Find n_i, ρ_i for ultra-relativistic particles. (hint: $|k| \gg m$, $m \ll T$) NB: for bosons $\mu \leq m$ (otherwise $f < 0$), while for fermions no restrictions, but observationally μ small.

4) Compare $\rho(a)$ with $\rho(T)$ for ultra-relativistic particles
5) Find n_i, ρ_i for non-relativistic particles (hint: $m \gg T$, $k \ll m$)

NB: $T \ll m_i - \mu_i$ (dilute system)

6) Compute ρ_{tot}. Which term dominates?

1) $E = |k| \longrightarrow p_i = g_i \int \frac{d^3k}{(2\pi)^3} \frac{E}{3} f_i(E)$

2) $\rho \propto a^{-3(1+w)} \longrightarrow w=0 : \rho \propto a^{-3}$

$w = \frac{1}{3} : \rho \propto a^{-4}$

3) $$f_i = \frac{1}{\exp (E-\mu)/T \pm 1}$$

$$f_i^B \approx \frac{1}{\exp K/T - 1} , \quad f_i^F \approx \frac{1}{\exp K/T + 1}$$

$$n_i \approx g_i \int_0^\infty \frac{4\pi K^2 dK}{(2\pi)^3} \frac{1}{\exp(K/T) \pm 1} =$$

$$= \frac{g_i}{2\pi^2} \int_0^\infty \frac{K^2}{\exp(K/T) \pm 1} dK$$

$$\int_0^\infty \frac{x^2}{\exp\left(\frac{x}{a}\right) - 1} dx = 2\zeta(3)\, a^3$$

$$\int_0^\infty \frac{x^2}{\exp\left(\frac{x}{a}\right) + 1} dx = \frac{3}{2}\zeta(3)\, a^3$$

≈ 1.2

$$= \frac{\zeta(3)}{\pi^2} g_i T^3 \begin{cases} 1 & (\text{bosons}) \\ 3/4 & (\text{fermions}) \end{cases}$$

$$\rho_i = \frac{g_i}{2\pi^2} \int_0^\infty \frac{K^3}{\exp(K/T) \pm 1} dK$$

$$\int_0^\infty \frac{x^3}{\exp\left(\frac{x}{a}\right) - 1} dx = \frac{\pi^4}{15} a^4$$

$$\int_0^\infty \frac{x^3}{\exp\left(\frac{x}{a}\right) + 1} dx = \frac{7\pi^4}{120} a^4$$

$$= \frac{\pi^2}{30} g_i T^4 \begin{cases} 1 & (\text{bosons}) \\ 7/8 & (\text{fermions}) \end{cases}$$

$$p_i = \frac{1}{3} \rho_i$$

4) $\begin{cases} \rho \propto a^{-4} \\ \rho \propto T^4 \end{cases} \longrightarrow a \propto T^{-1}$ for relativistic particles

$$E = \sqrt{k^2 + m^2} = m\sqrt{1 + \frac{k^2}{m^2}} \approx m\left(1 + \frac{1}{2}\frac{k^2}{m^2}\right) = m + \frac{k^2}{2m}$$

5) $$f_i = \frac{1}{\exp(E-\mu)/T \pm 1} \approx \exp(\mu_i - E)/T =$$

$$= \exp\left(\mu_i - m_i - \frac{k^2}{2m_i}\right)/T$$

$$n_i = \underbrace{\frac{g_i}{2\pi^2}\int_0^\infty \frac{k^2}{\exp\left(m_i + \frac{k^2}{2m_i} - \mu_i\right)/T}\,dk}_{\int_0^\infty x^2 e^{-x^2/2}\,dx = \sqrt{\frac{\pi}{2}}}$$

$$\boxed{\int_0^\infty k^2 e^{(m_i-\mu_i)/T} e^{-k^2/2m_iT}\,dk \xrightarrow[\frac{k}{\sqrt{m_iT}} \to x]{} \int_0^\infty (m_iT)^{3/2} e^{-(m_i-\mu_i)/T} e^{-x^2/2}\,dx = (m_iT)^{3/2} e^{-(m_i-\mu_i)/T}\sqrt{\frac{\pi}{2}}}$$

$$\rho_i = \frac{g_i}{2\pi^2}\int_0^\infty \frac{m_i k^2}{\exp(\;)}\,dk + \underbrace{\frac{g_i}{2\pi^2}\int_0^\infty \frac{k^4/2m_i}{\exp(\;)}\,dk}_{\int_0^\infty x^4 e^{-x^2/2}\,dx = 3\sqrt{\frac{\pi}{2}}}$$

$$\boxed{\frac{e^{-(m_i-\mu_i)/T}}{2m_i}\int_0^\infty k^4 e^{-k^2/2m_iT}\,dk \xrightarrow[\frac{k}{\sqrt{m_iT}} \to x]{} \frac{e^{-(m_i-\mu_i)/T}}{2m_i}\int_0^\infty (m_iT)^{5/2} e^{-x^2/2} x^4\,dx = \frac{1}{2}T(m_iT)^{3/2} e^{-(m_i-\mu_i)/T}\int_0^\infty x^4 e^{-x^2/2}\,dx = \frac{3}{2}Tm_i}$$

$$\rightarrow \rho_i = n_i\left(m_i + \frac{3}{2}T\right)$$

6) $$\rho_{tot} = \rho_{rel} + \rho_{non-rel} \to \sim e^{-(m_i-\mu_i)/T}$$

$$m \gg T \longrightarrow \sim 0$$

EXERCISE 8

Derive the required field range $\phi_i - \phi_f$ to get $N=60$, with a linear potential $V(\phi) = \phi$

$$N = \int_{t_i}^{t_f} H(t)\,dt = \int_{t_i}^{t_f} H(t) \frac{dt}{d\phi}\,d\phi = \int_{\phi_i}^{\phi_f} \frac{H}{\dot\phi}\,d\phi$$

$$\begin{cases} \dot\phi \simeq -\dfrac{V'}{3H} \\ H^2 \simeq \dfrac{V}{3M_{Pl}^2} \end{cases} \longrightarrow N = -\int_{\phi_i}^{\phi_f} \frac{3H^2}{V'}\,d\phi = -\frac{1}{M_{Pl}^2}\int_{\phi_i}^{\phi_f} \frac{V}{V'}\,d\phi$$

$$\xrightarrow[V=\phi]{} N = -\frac{1}{M_{Pl}^2}\int_{\phi_i}^{\phi_f} \phi\,d\phi = \frac{1}{2M_{Pl}^2}\left(\phi_i^2 - \phi_f^2\right)$$

$$\phi_f = ? \longrightarrow \varepsilon\Big|_{end} = \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right)^2\Bigg|_{end} = \frac{M_{Pl}^2}{2}\frac{1}{\phi_f^2} = 1$$

From Friedmann equations $\ddot a > 0$ when $\varepsilon < 1$.
$\Longrightarrow \varepsilon = 1$ end !

$$\longrightarrow \phi_f^2 = \frac{M_{Pl}^2}{2}$$

$$\longrightarrow \frac{1}{2M_{Pl}^2}\left(\phi_i^2 - \frac{M_{Pl}^2}{2}\right) = 60$$

$$\longrightarrow \phi_i^2 \simeq 120.5\, M_{Pl}^2$$

EXERCISE 9

The dimensionless power spectrum $\Delta^2_{\zeta}(k)$ is defined as:

$$\Delta^2_{\zeta}(k) \equiv \frac{k^3}{2\pi^2} P_{\zeta}(k)$$

where:

$$\langle \zeta_{\bar{k}} \zeta_{\bar{k}'} \rangle = (2\pi)^3 \delta^{(3)}(\bar{k}+\bar{k}') P_{\zeta}(k)$$

Knowing that $\Delta^2_{\zeta}(k) \simeq 2 \times 10^{-9}$, which is the value of the mass m that the inflaton field has to have, considering $V = \frac{1}{2} m^2 \phi^2$?
(NB: evaluate it when the largest λ leaves the horizon, i.e. beginning of inflation)

Since ζ approaches a constant on super-horizon scales, we can evaluate the spectrum at horizon crossing and this determines the future spectrum until a given fluctuation mode re-enters the horizon.
The dimensionless scalar power spectrum has then the following form:

$$\Delta^2_{\zeta}(k) = \frac{1}{8\pi^2} \frac{H^2}{M_p^2} \frac{1}{\varepsilon} \Bigg|_{k=aH}$$

$$H^2 = \frac{V}{3M_{pl}^2} \longrightarrow \Delta^2_{\zeta}(k) \simeq \frac{1}{12\pi^2 M_p^6} \frac{V^3}{V'^2} \Bigg|_{k=aH}$$

For a potential $V = \frac{1}{2} m^2 \phi^2$ we then have:

$$\Delta^2_{\zeta}(k) = \frac{1}{96\pi^2 M_p^6} m^2 \phi^4 \Bigg|_{k=aH}$$

When largest λ leaves the horizon we are at the beginning of inflation!
Smaller λ need more time to grow till H^{-1}.
When $\lambda_{phys} \sim H^{-1}$, $a\lambda_{com} \sim H^{-1} \longrightarrow \frac{a}{k} \sim H^{-1} \longrightarrow k \sim aH$

ϕ_i is computed as in the previous exercise:

$$\phi_i \simeq 15 \, M_{Pl}$$

$$\longrightarrow m \simeq 6.12 \times 10^{-6} \, M_{Pl}$$

THEORETICAL PARENTHESIS

$$\begin{cases} ds^2 = a^2(\tau)\left[-d\tau^2 + \gamma_{ij}\,dx^i dx^j\right] \\ ds^2 = a^2(\tau)\left[-(1+2\Phi)\,d\tau^2 + 2B_i\,dx^i d\tau + (\gamma_{ij} + E_{ij})\,dx^i dx^j\right] \end{cases}$$

Euclidean 3d metric

SVT decomposition $\longrightarrow$ transverse and traceless tensor h_{ij} (gravitational waves)

$h_{ij} \longrightarrow$ 2 polarizations, which evolve independently as 2 scalar fields

$$\begin{cases} P_\zeta(k) = \dfrac{1}{2\varepsilon M_{Pl}^2}\left(\dfrac{H_*}{2\pi}\right)^2 \left(\dfrac{k}{k_*}\right)^{3-2\nu} = A_s\left(\dfrac{k}{k_*}\right)^{n_s - 1} \\ A_s = \dfrac{H_*^2}{8\pi^2\varepsilon M_{Pl}^2}, \qquad n_s = 1 - 6\varepsilon_v + 2\eta_v \end{cases}$$

$$P_h(k) = A_T\left(\frac{k}{k_*}\right)^{n_T}, \qquad n_T = -2\varepsilon_v$$

Scale invariance: $n_s = 1$, $n_T = 0$

$$r = \frac{A_T}{A_s} = \frac{8M_{Pl}^2}{\phi^2} \approx 16\varepsilon$$

Exercise 10

Find the predictions for r and n_s of the potential $V = \phi$.

$$\begin{cases} n_s - 1 = -6\varepsilon + 2\eta \\ r = 16\varepsilon \end{cases}$$

$$\varepsilon = \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2, \qquad \eta = M_P^2 \frac{V''}{V}$$

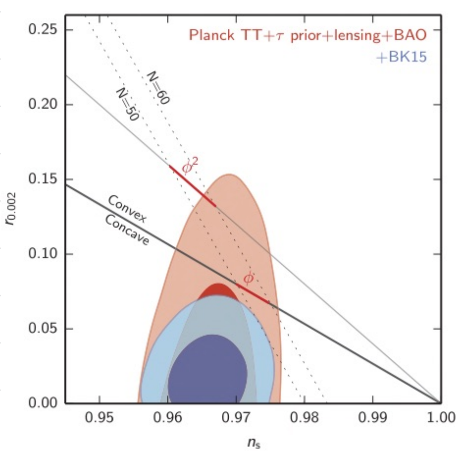
With this potential:

$$\varepsilon = \frac{M_P^2}{2\phi^2}, \qquad \eta = 0$$

$$\longrightarrow \quad n_s = 1 - 3\frac{M_P^2}{\phi^2}, \qquad r = \frac{8M_P^2}{\phi^2}$$

Evaluating at horizon crossing for the largest wavelength: $\phi_i \simeq 11\, M_{Pl}$

$$\longrightarrow \quad n_s \simeq 0{,}975, \qquad r \simeq 0{,}066$$



Looking at this plot (10.1103/PhysRevLett.121.221301) we see that these values for n_s and r are compatible with the current observations, except for the BK15 result, within 2σ. We conclude that $V \propto \phi$ is a viable model of the observed universe.

EXERCISE 11

Compute the recombination temperature, knowing that it is defined when 90% of protons are "caught" into hydrogen atoms, i.e we have 10% free electrons.

$$p + e^- \longleftrightarrow H + \gamma$$

When $T < B_H = 13.6$ eV electrons get trapped. Start from the Boltzmann equation

$$\frac{dn_p}{dt} + 3Hn_p = n_p^{eq} n_e^{eq} \langle \sigma v \rangle \left(\frac{n_H n_\gamma}{n_H^{eq} n_\gamma^{eq}} - \frac{n_p n_e}{n_p^{eq} n_e^{eq}} \right)$$

If reaction rate $\Gamma \gg H \longrightarrow$ rhs much larger than lhs. Only way is the individual terms in $(\cdot)$ cancel separately:

SAHA EQUATION $\longleftarrow$ $$\frac{n_H n_\gamma}{(n_H n_\gamma)_{eq}} = \frac{n_p n_e}{(n_p n_e)_{eq}} \iff \mu_1 + \mu_2 = \mu_3 + \mu_4$$ (chemical equilibrium)

Photons at equilibrium $\longrightarrow n_\gamma = n_\gamma^{eq}$

$$\longrightarrow \frac{n_e n_p}{n_H} = \frac{n_e^{eq} n_p^{eq}}{n_H^{eq}}$$

Free electrons fraction: $$X_e = \frac{n_e}{n_e + n_H} = \frac{n_p}{n_p + n_H}$$

$$n_i^{eq} = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T} \right), \qquad g_p = g_e = 2$$

NB: $m_e \sim 0.5$ MeV $\longrightarrow$ non-relativistic
$m_p \sim 1$ GeV $\longrightarrow$ non-relativistic

$$\frac{X_e^2}{1-X_e} = \frac{n_e^2}{(n_e+n_H)^2}\,\frac{n_e+n_H}{2n_e+n_H} =$$

$$= \frac{1}{n_e+n_H}\cdot\frac{n_e^2}{n_H} =$$

$$= \frac{n_e n_p}{n_H}\,\frac{1}{n_e+n_H} =$$

$$= \frac{1}{n_e+n_H}\left(\frac{m_e T}{2\pi}\right)^{3/2} \exp\left[-(m_e+m_p-m_H)/T\right]$$

$= n_p + n_H =$ baryon density (without helium nuclei)

$= 0.76\,\boxed{n_B} = 0.76\,\boxed{\eta\, n_\gamma}$

$\eta =$ baryon to photon ratio

$X_H \approx 0.76$

$X_{He} \approx 0.24$ (at the end of BBN)

$\eta = 6\times10^{-10}$

$n_\gamma^{eq} = 2\cdot\frac{\zeta(3)}{\pi^2}\,T^3$

$$\longrightarrow \frac{X_e^2}{1-X_e} = \frac{\sqrt{\pi}}{0.76\cdot4\sqrt{2}\,\zeta(3)\,\eta}\left(\frac{m_e}{T}\right)^{3/2} e^{-B_H/T}$$

2 regimes: (1) $T \gtrsim B_H$

(2) $T < B_H$

$X_e = 0.1 \implies$ (2) $\quad X_e \ll 1$

$$X_e(T) \simeq \left[\frac{\sqrt{\pi}}{0.76\cdot4\sqrt{2}\cdot\zeta(3)}\,\frac{1}{\eta}\left(\frac{m_e}{T}\right)^{3/2} e^{-B_H/T}\right]^{1/2} = 0.1$$

$$\implies T_{rec} \approx 0.3\,\text{eV} \ll B_H = 13.6\,\text{eV}$$