BHPT

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Outline

(Luca Santoni – IPhT lectures on BHPT
S. Chandrasekhar – Mathematical theory of BHs)

1) Perturbations of Schwarzschild

- Tidal deformations and quasi-normal modes
- Metric perturbations
- Zerilli and RW eq
- Chandrasekhar duality isospectrality

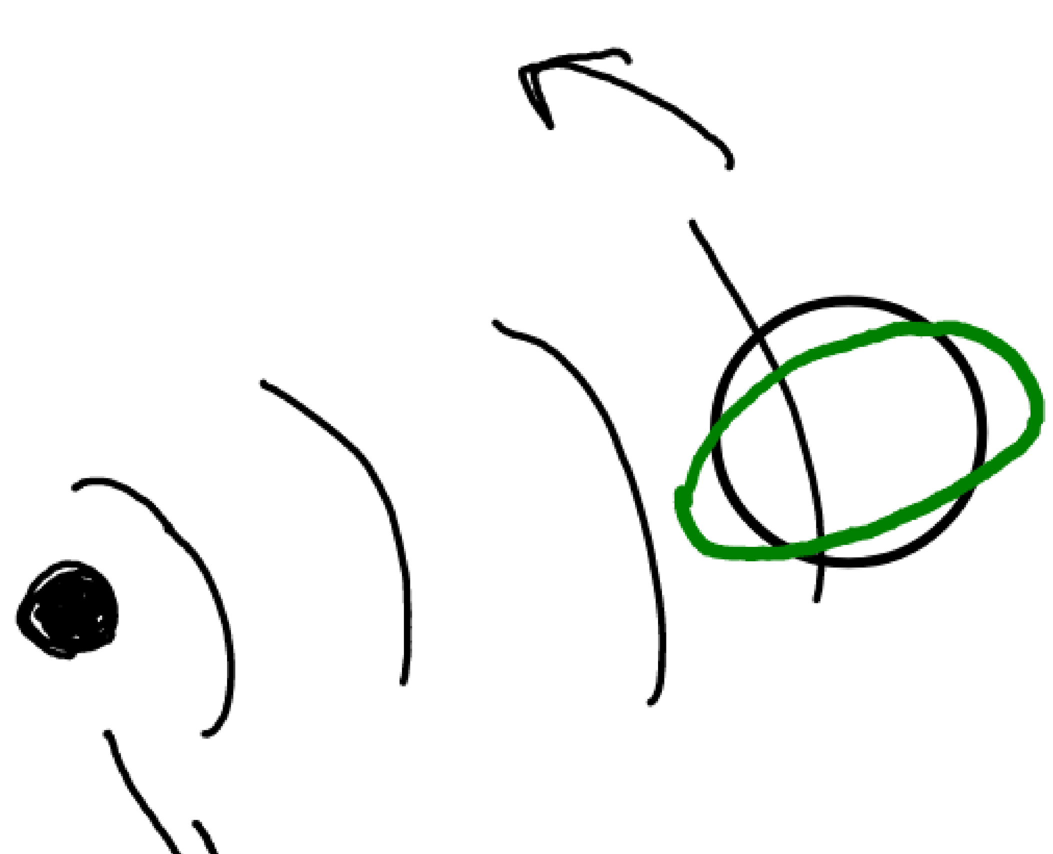
2) QNMs

- Boundary conds Green's function
- not complete
- Eikonal approx
- Teukolsky eq

3) Tides

- Newtonian
- Generalise to GR
- Mysterious zeros
- Symmetries of BHPT
- Nonlinearities

Physical situations we care about?



Tidal deformations

$\frac{R}{\lambda} \ll 1$

Quasi-normal modes (QNMs)

Damped oscilations of a BH

Schwarzschild metric in Schwarzschild coordinates

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$f(r) = 1 - \frac{r_s}{r} \,, \quad r_s \equiv 2GM$$

*) Sch background has

- Spherical symmetry $SO(3)$ $\to$ 3KV
- Time translations (time independent) $\to$ 1KV

will simplify studying of perturbations (which break these symms)

*) $$g_{\mu\nu} = \underbrace{g^{sch}_{\mu\nu}}_{\text{Background}} + \underbrace{h_{\mu\nu}}_{\text{Perturbation}} \,, \quad \| h_{\mu\nu} \| << \| g^{sch}_{\mu\nu} \|$$

~ Gravitons on top of Sch background

$$R_{\mu\nu}[g = g^{sch} + h] = 0$$

$$= \underbrace{R_{\mu\nu}[g^{sch}]}_{=0} + \boxed{\hat{\mathcal{D}}\, h_{\mu\nu} = 0} \quad \sim \quad \partial\partial h = 0$$

· we expect 2 degrees of freedom

* Metric parametrization

$$\varphi(x) = \int \frac{d\omega}{2\pi} \sum_{\ell m} \underbrace{e^{-i\omega t} \overset{\text{spherical harmonics}}{\overbrace{Y_{\ell m}(\theta,\varphi)}} \varphi_{\ell m}(r,\omega)}_{\text{expansion adapted to symmetries of the background}}$$

$$h_{\mu\nu} = \int \frac{d\omega}{2\pi} e^{-i\omega t} \sum_{\ell m} (\ ?\ )$$

• vector $v_\mu(x) = (v_t(x), v_i(x))$

$$= (v_t(x), v_r(x), \underbrace{v_\theta(x), v_\varphi(x)}_{\equiv v_A(x)})$$

$v_A(x)$: vector on a sphere, $A \in \{\theta, \varphi\}$

ex • $v_A = \partial_A V(x)$, with $V(x)$ a scalar: $V(x) = \sum_{\ell m} Y_{\ell m}(\theta,\varphi) V_{\ell m}$

$$\Rightarrow v_A = \sum_{\ell m} V_{\ell m} \boxed{\partial_A Y_{\ell m}(\theta,\varphi)}$$

ex • $w_A = \varepsilon_{AB} \partial_B W = \sum_{\ell m} W_{\ell m} \boxed{\varepsilon_{AB} \partial_B Y_{\ell m}}$

ε_{AB} – Levi-Civita tensor (on a sphere)

$$\varepsilon_{AB} = -\varepsilon_{BA} \quad , \quad \varepsilon_{\theta\varphi} = (\det \gamma_{AB})^{1/2}$$

$$\gamma_{AB}\, dx^A dx^B = d\theta^2 + \sin^2\theta\, d\varphi^2$$

metric on a sphere

<u>The most general vector</u> $\quad u_A = \sum_{lm} V_{lm}\, \partial_A Y_{lm} + W_{lm}\, \varepsilon_{AB}\, \partial^B Y_{lm}$

Tensor?

scalars: $h_{tt}, h_{tr}, h_{rr}, \gamma^{AB} h_{AB}$ (4)

vectors: h_{tA}, h_{rA}

tensors: h_{AB} ?

$(\;)_{AB}$?

- $\partial_A \partial_B Y_{lm}(\theta,\varphi) \rightarrow \nabla_A \nabla_B Y_{lm}(\theta,\varphi)$

covariant derivative on a sphere (more convenient)

- $\varepsilon_{AC} \nabla_B \nabla^C Y_{lm}(\theta,\varphi) + \varepsilon_{BC} \nabla_B \nabla^C Y_{lm}(\theta,\varphi)$

scalars + vectors + tensors = 10 scalars
(4) (2×2) (2)

$$\varepsilon_{AB} \xrightarrow{\vec{x} \to -\vec{x}} -\varepsilon_{AB}$$

$$\Rightarrow \quad \nabla_A Y_{\ell m} \to (-1)^\ell \partial_A Y_{\ell m} \quad \leftarrow \text{even}$$

under parity

$$\varepsilon_{AB} \nabla^B Y_{\ell m} \to (-1)^{\ell+1} \varepsilon_{AB} \nabla^B Y_{\ell m} \quad \swarrow \text{odd}$$

$$\to \quad h_{\mu\nu} = h^+_{\mu\nu} + h^-_{\mu\nu}$$

<u>even pert</u>: $h^+_{\mu\nu} = \begin{pmatrix} f(r) H_0 & H_1 & \nabla_A \alpha \\ H_1 & \frac{1}{f(r)} H_2 & \nabla_A \beta \\ (\nabla_A \alpha)^T & (\nabla_A \beta)^T & \nabla_A \nabla_B G + r^2 \gamma_{AB} \end{pmatrix}$

<u>odd pert</u>: $h^-_{\mu\nu} = \begin{pmatrix} 0 & 0 & \varepsilon_{AB} \nabla^B h_0 \\ 0 & 0 & \varepsilon_{AB} \nabla^B h_1 \\ * & * & \\ * & * & (\quad)_{AB} \end{pmatrix}$

$$\hookrightarrow \left(\varepsilon_{AC} \nabla^C \nabla_B + \varepsilon_{BC} \nabla^C \nabla_A \right) h_2$$

$H_0, H_1, H_2, k, G, \alpha, \beta$,

h_0, h_1, h_2 are all scalar functions

$\to$ we expand them in $e^{-i\omega t} Y_{\ell m} \ldots$

* Gauge fixing

(see 2010.00593 v3)

flat background $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

curved —"— $\delta h_{\mu\nu} = \nabla^{Sch}_\mu \xi_\nu + \nabla^{Sch}_\nu \xi_\mu = \ldots =$

$$= \xi^\rho \partial_\rho g^{Sch}_{\mu\nu} + g^{Sch}_{\mu\rho} \partial_\nu \xi^\rho + g^{Sch}_{\nu\rho} \partial_\mu \xi^\rho$$

$$\xi_\mu = \left(f(r)\xi_0, \frac{\xi_r}{f(r)}, D_A \xi_S + \varepsilon_{AB} D^B \xi_V \right)$$

$\Rightarrow$ $\delta G = \frac{2\xi_S}{r^2}$ $\qquad$ $\delta h_2 = \frac{2}{r}\xi_V$

$\delta \alpha = \dot{\xi}_S + f\xi_0$ $\qquad$ $\delta H_0 = \ldots \quad \ldots$

$\delta \beta = -2\frac{\xi_S}{r} + \xi_S' + \frac{\xi_1}{f}$

$\rightarrow$ $\boxed{\alpha = \beta = G = 0 \ \& \ h_2 = 0}$

Regge-Wheeler gauge

$$\frac{d^2\psi_{Z/RW}}{dr_*^2} + \left(\omega^2 - V_{Z/RW}(r) \right) \psi_{Z/RW} = 0$$

$$\delta\psi_Z = \left(\frac{\partial}{\partial r_*} - W(r) \right) \psi_{RW}$$

$$\delta\psi_{RW} = \left(\frac{\partial}{\partial r_*} + W(r) \right) \psi_Z$$

$$W = \frac{3 r_s (r_s - r)}{r^2 (3 r_s + 2\lambda r)} - \frac{2\lambda(\lambda+1)}{3 r_s} \qquad \lambda \equiv \frac{1}{2}(L-1)(L+2)$$

Chandrasekhar duality

(2310.04502)

- Can be formulated as an off-shell theorem of the S_{EH}, but non-local => conservation law
- Electric-magnetic duality („$\vec{E} \leftrightarrow \pm \vec{B}$")
- Generalized Darboux transformation
- appears in Supersymmetric Quantum Mechanics
- KdV eq. $\partial_t u - 6u\partial_x u + \partial_x^3 u = 0$

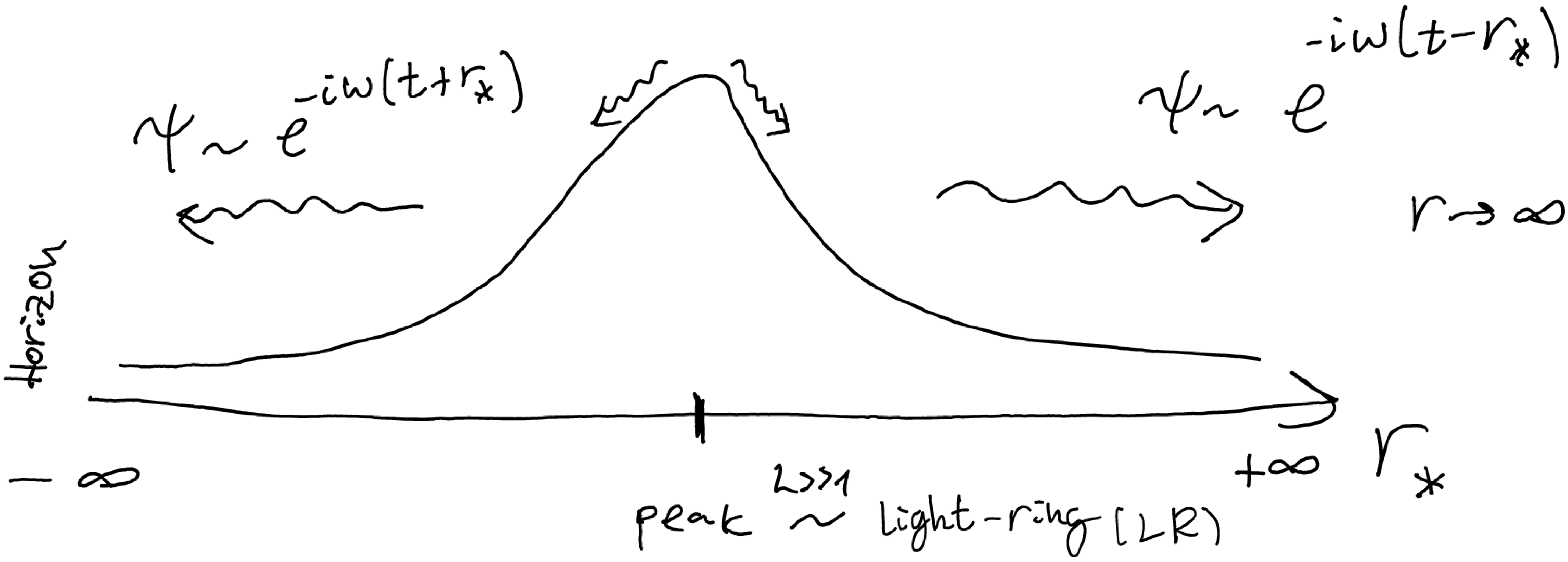
Physically

- equality of QNM freqs (i.e. isospectrality)
- — " — of Love Numbers
- — " — transmission and reflection coeffs in scattering

Quasi-Normal Modes

- perturbations propagate away from the BH (nothing incoming from far away)

- nothing leaves a BH (at least classical



- We expect $\mathrm{Im}\,\omega_{n\ell} \neq 0$ (dissipation due to horizon)

- To solve this eigenvalue problem exactly we need numerical methods

- For large $L \gg 1 \to$ geometric optics limit $\to$ WKB
  $\to$ QNMs are unstable gravitons spilling from the light ring

$$\omega \overset{L \gg 1}{\simeq} \mathrm{Re}(\omega) + i\,\mathrm{Im}(\omega)$$

$\mathrm{Re}(\omega) \sim \frac{1}{\text{Period of LR orbit}}$ (stable in θ dir.)

$\mathrm{Im}(\omega) \sim$ Lyapunov rate in r_* direction

$$\omega_{WKB} \approx \left(l+\frac{1}{2}\right)\omega_{orb} - i\lambda_L\left(n+\frac{1}{2}\right)$$

important minus!

$$\lambda_L = \omega_{orb} = \frac{1}{3\sqrt{3}GM}$$

$$\psi \sim e^{-i\omega t} \sim e^{+\mathrm{Im}(\omega)t}\, e^{-i\,\mathrm{Re}(\omega)t}$$

$\mathrm{Im}(\omega) < 0 \Rightarrow$ stability
(under linear perts)

Greens function

$$x \equiv r_*$$

$$\underbrace{\left(\partial_t^2 - \partial_x^2 + V(x)\right)}_{\equiv \mathcal{L}_x} \psi(x) = J(x)$$

if we add a source of perturbations

$$\mathcal{L}_x\, G(t-t', x, x') = \delta(\bar{x}-\bar{x}')\,\delta(t-t')$$

Retarded / causal boundary cond. $G(t-t'<0, x, x') = 0$

future doesn't influence the past

$$\rightarrow \quad \mathcal{L}_x^{(\omega)}\, G(\omega, x, x') = \delta(\bar{x}-\bar{x}')$$

$$G(\omega) = \int_{-\infty}^{\infty} dt\, G(t, x, x')\, e^{+i\omega t} \qquad \omega \in \mathbb{C}$$

$$e^{-\mathrm{Im}(\omega)t}\, e^{i\,\mathrm{Re}(\omega)t} \Rightarrow$$

$\Rightarrow$ $\left[G(\omega) \text{ is analytic in uper half complex plan } (\mathrm{Im}(\omega) > 0) \right]$ (*)

$$L_x \, G(\omega, x, x') \overset{x \neq x'}{=} 0$$

$$\to \quad G(\omega, x, x') = \#(x') \, \Psi_+ + \#'(x') \, \Psi_-$$

two independent solutions of linear equation

we choose them such that

$$\Psi_+(\omega, x \to -\infty) \xrightarrow{\text{horizon}} e^{-i\omega x}$$

$$\Psi_-(\omega, x \to +\infty) \xrightarrow{\text{far}} e^{i\omega x}$$

$$(*) \Rightarrow G(\omega, x, x') = \begin{cases} \propto \Psi_+(x) A(x') \propto e^{-i\omega x} & \text{for } x < x' \\ \propto \Psi_-(x) B(x') \propto e^{+i\omega x} & \text{for } x > x' \end{cases}$$

$\left(\text{otherwise } G \underset{x \to \infty}{\sim} e^{|\mathrm{Im}(\omega)| x} \underset{\to \infty}{} \right)$

- Impose continuity of G $(x<x', x>x')$ and jump of $\partial_x G$ $\Rightarrow$

$$\Rightarrow \boxed{G(\omega, x, x') = \frac{1}{W(\omega)} \left[\Psi_+(\omega, x') \Psi_-(\omega, x) \Theta(x - x') + \Psi_-(\omega, x') \Psi_+(\omega, x) \Theta(x' - x) \right]}$$

Wronskian

$$\Theta(y) \equiv \begin{cases} 0, & y < 0 \\ 1, & y \geq 0 \end{cases}$$

(„Theta function")

$$\boxed{G(\omega, x, x') = \frac{1}{W(\omega)}\left[\Psi_+(\omega, x')\,\Psi(\omega, x)\,\theta(x-x') + \Psi_-(\omega, x')\,\Psi_+(\omega, x)\,\theta(x-x')\right]}$$

$$W \equiv \Psi_-(x')\,\partial_{x'}\Psi_+(x') - \Psi_+(x')\,\partial_{x'}\Psi_-(x') = W(\omega)$$

$$\partial_{x'} W = 0$$

(from eq. of motion)

if we find ω_* such that $W(\omega_*) = 0$

$\Rightarrow \Psi_+(\omega_*, x) \propto \Psi_-(\omega_*, x)$

$\Rightarrow \Psi_\pm(\omega_*, x)$ satisfy both QNM bound. conds. !

$\Rightarrow$ $\boxed{\text{QNM frequencies are poles of Greens function}}$

(* for some special ω_* for which $W(\omega_*) = 0$
it happens that $G(\omega_*) = \frac{0}{0} \rightarrow$ pole skipping
interesting for quantum gravity considerations
i.e. holography)

Completeness ?

Can we write the most general perturbation as

$$\psi(x,t) = \int G(x,x')\,J(x')\,dx' \overset{?}{=} \sum_{q\in\{QNMs\}} c_q\, e^{-i\omega_q t}\, \psi_+(x,\omega_q)$$

Like we do for bunch of harmonic oscillators or for textbook QM potentials ?

i.e. in QM $\quad G(\omega) = \sum_n \dfrac{\psi_n \psi_n^*}{\omega - E_n}$ w/

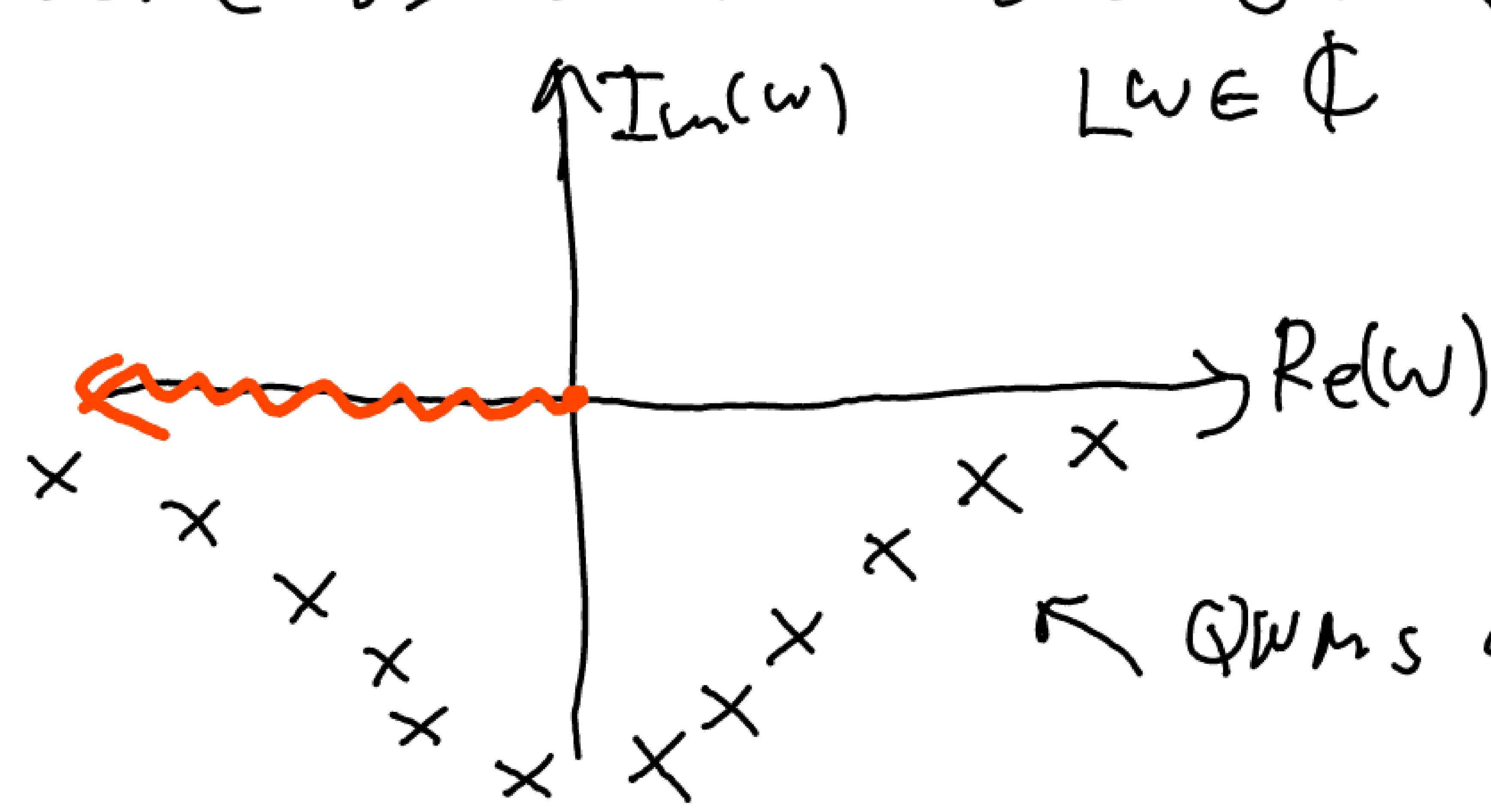
QNMs are not complete

- mathematical subtleties $\left(\psi_+(x) \xrightarrow[\substack{x\to\infty \\ \omega \text{ fixed}}]{} \infty \right)$
- $G \xrightarrow[x\to-\infty]{} \sim e^{-\#|x|}$ but $G \xrightarrow[x\to+\infty]{} \sim \dfrac{1}{x^2}$

  this slow fall-off is due to long range nature of gravity

- there is a branch cut! $\quad \omega \in \mathbb{C}$ $\quad$ => eigenmode expansion cannot be true

G :



QNMs are poles

Isospectrality

- Chandra's duality maps $Z \leftrightarrow RW$ equation (i.e. dynamics) but we need to check boundary conds.

$$\Psi_{RW} \propto \left(\frac{\partial}{\partial r_*} + W(r)\right)\Psi_Z$$

$$W = \frac{3r_s(r_s - r)}{r^2(3r_s + 2\lambda r)} - \frac{2\lambda(\lambda+1)}{3r_s}$$

- $r \to \infty$ $\quad \Psi_Z \sim e^{-i\omega(t - r_*)}$ (purely outgoing at infinity)

$$\Psi_{RW} \sim \Big(\underbrace{\frac{\partial}{\partial r_*}}_{\text{const}} + \underbrace{W(r)}_{\text{const}}\Big) e^{-i\omega(t-r_*)} \sim \text{const} \cdot e^{-i\omega(t-r_*)} \qquad ✓$$

- $r \to r_s$ $\quad \Psi_Z \sim e^{-i\omega(t + r_*)}$ (purely ingoing at horizon)

$$\left(\frac{\partial}{\partial r_*} + W(r)\right) e^{-i\omega(t+r_*)} \sim \text{const}\ e^{-i\omega(t+r_*)} \qquad ✓$$

$\Rightarrow$ $\Psi_Z(r_*, \omega_n)$ is a QNM $\Rightarrow (\partial_* + W(r))\Psi_Z(r_*, \omega_n) \sim$

$\sim \Psi_{RW}(r_*, \omega_n)$ is QNM too

- true for Kerr as well, but not in $D \neq 4$ nor in AdS nor Beyond GR

Perturbations of Kerr

[Chandrasekhar Mathematical Theory of BHs]

- $g_{\mu\nu} = g^{Kerr}_{\mu\nu} + h_{\mu\nu} \xrightarrow{SVT}$ some scalars $\rightarrow \dots$

$$K(\vec{x},t) = \sum_m \int \frac{d\omega}{2\pi} e^{-im\varphi} e^{-i\omega t} K_m(\omega,\theta,r)$$

with metric pert. we <u>cannot</u> separate eqs. in θ and r!

is there a way around? – YES!

*) <u>Weyl tensor</u> (remove traces of Riemann)

$$C_{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha\beta} + \frac{R}{6}\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right) +$$
$$+ \frac{1}{2}\left(R_{\mu\beta}g_{\nu\alpha} - R_{\mu\alpha}g_{\nu\beta} + R_{\nu\alpha}g_{\mu\beta} - R_{\nu\beta}g_{\mu\alpha}\right)$$

- same symmetries as Riemann
- $20 - 10 = 10$ components $= 5$ even $+ 5$ odd

these contain physical propagating d.o.f. of GWs

*) <u>Newman-Penrose formalism</u>

- take <u>null</u> tetrads $\{ l^\mu, n^\mu, m^\mu, \bar{m}^\mu \}$

complex conj.

- contract every tensor with

$\rightarrow$ work with scalars only

- we choose the tetrad such that

$$g_{\mu\nu} \ell^\mu q^\nu = \ell \cdot q = -1 \qquad m \bar{m} = 1 \qquad \text{and all others} = 0$$

- In Boyer-Lindquist coords convenient choice is Kinnersley tetrad:

$$\ell^\mu = \frac{1}{\Delta}(r^2+a^2, \Delta, 0, a)$$

$$q^\mu = \frac{1}{2\rho^2}(r^2+a^2, -\Delta, 0, a)$$

$$m^\mu = \frac{1}{\sqrt{2}} \frac{1}{r+ia\cos\theta}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right)$$

- $C_{\mu\nu\alpha\beta} \leftrightarrow$ 5 Complex (Weyl) scalars

$$\Psi_0 \equiv C_{\mu\nu\alpha\beta} \ell^\mu m^\nu \ell^\alpha m^\beta$$

$$\Psi_1 \equiv \dots$$

$$\dots \quad \Psi_4 \equiv \dots$$

- $R_{\mu\nu} = 0 \Rightarrow$ EOM for $\Psi_{0,\dots,4}$

- perturbations: $\Psi_i = \Psi_i^{\text{Kerr}} + \delta\Psi_i$

- cover all spins at once $|S| = 0, \pm 1, \pm 2$

perturbations: scalar, electromagnetism, gravity

$$\delta\Psi_2 = \int \frac{d\omega}{2\pi} \sum_{\ell m} e^{-i\omega t + im\varphi} S^{(s)}_{\ell m}(\theta) R^{(s)}_{\ell m}(r)$$

- $S^{(s)}_{\ell m}(\theta)$ - spin s spheroidal harmonics

$$S^{(s)}_{\ell m} \xrightarrow{a\to 0} Y^{(s)}_{\ell m}(\theta) \xrightarrow{s=0} Y_{\ell m}(\theta)$$

good old scalar spherical harmonics

Teukolsky equation:

$$\boxed{\Delta^{-s}\partial_r\left(\Delta^{s+1}\partial_r R^{(s)}\right) + \left[\frac{K^2(\omega) - is(2r-r_s)K(\omega)}{\Delta} + 4is\omega r - \lambda^{(s)}_{\ell m}(\omega)\right]R^{(s)} = 0}$$

$$K(\omega) \equiv (r^2+a^2)\omega - am \qquad \Delta \equiv r^2 - r r_s + a^2$$

$$\lambda^{(s)}_{\ell m} \equiv A^{(s)}_{\ell m}(\omega) + a^2\omega^2 - 2ma\omega$$

separation constant (fixed in eigen-eq for $S^{(s)}_{\ell m}(\theta)$ by regularity $\theta\in[0,\pi]$)

(no closed form expr. numerics)

- $a\to 0$ we don't recover RW/Z eq, but Bardeen-Press eq
- Teukolsky-Starobinsky identities [$+s \leftrightarrow -s$)
- ladder operators in spin for $\omega = 0$ $\delta\Psi^{(s\pm 1)} = E^{\pm}_{(s)}\delta\Psi^{(s)}$
- not so simple equation (CFT, SUSY, Seiberg-Witten theory)

Why did this work?

- Newman-Penrose formalism is particularly well adapted to BH spacetimes
- Petrov algebraic classification (see Chandra's book)
  - Use Lorentz transfs. to set as many $\Psi_i = 0$
  - "Hidden symmetries"
  - BHs are type D spacetimes, i.e. only 1 non-trivial Ψ
  - useful in other situations too

(*) Mano-Suzuki-Takasugi (MST) method

- for solving Teukolsky in $GM\omega \ll 1$
- based on asymptotic matching technique:

  far away $R \sim \sum_n$ Coulomb wavefunctions

  close to horizon $R \sim \sum_n$ Hypergeom. functions

Tidal deformations of BHs

compact objects deform each other gravitationaly

(*) Tides in Newtonian gravity

static regime

$$\nabla^2 \phi = 4\pi G \rho(\vec{x})$$

in isolation in vacuum

$$\phi_0 = -\frac{GM}{r}$$

in vacuum $\nabla^2 \phi = 0 \rightarrow \phi = \phi_0 + \sum_{\ell \geq 2}^{\infty} \mathcal{E}_{\ell m} \Big(r^\ell + \frac{\lambda_\ell}{r^{\ell+1}} \Big) Y_{\ell m}$

tidal field

"response" field from the object

. dimensional analysis $\boxed{\lambda_\ell \sim \# R^{2\ell+1}}$

$\# \sim O(1)$

λ_ℓ – Love numbers (of multipole ℓ)

. they are properties of the compact object

. like electric polarization of a medium

(*) How about GR

- Naive proposal $g_{00} = -1 + 2\phi$

$$\frac{1+g_{00}}{2} = -\frac{GM}{r} + \sum_{\substack{\ell m \\ \ell \geq 2}} \mathcal{E}_{\ell m}\left(r^\ell + \frac{\lambda_\ell}{r^{\ell+1}}\right)Y_{\ell m} + \underbrace{\cdots}_{\text{GR nonlinearities}} + \cdots$$

Subtleties of GR

Definition above suffers from at least two things

- r_s corrections to tidal field

$$\mathcal{E}_{\ell m} r^\ell \left(1 + \# \frac{r_s}{r} + \# \left(\frac{r_s}{r}\right)^2 + \cdots\right) = \cdots + \mathcal{E}_{\ell m} \frac{r_s^{2\ell+1}}{r^{\ell+1}} + \cdots$$

degenerate with $\mathcal{E}\,\frac{\lambda_\ell}{r^{\ell+1}}$

- g_{00} is not gauge invariant

$r \to \tilde{r}(r)$ changes the value of coeffs in expansion

Clean and practical framework for tides is

Worldline Effective Field Theory

(From „far away" Compact object = point particle + finite size corrections (e.g. Love #s)

more on this in lectures of Rafael)

• In GR there are electric (even) and magnetic (odd) Love numbers λ_ℓ^E λ_ℓ^B

• in $D=4$ pure GR (asymp. flat) $\boxed{\text{Chandra} \Rightarrow \lambda_\ell^E = \lambda_\ell^B}$

• by dimensional analysis, for BHs we expect $\lambda_\ell \sim \mathcal{O}(1) \times r_s^{2\ell+1}$

Remarkably all $\lambda_\ell^{BH} = 0$ in $D=4$ pure GR

(in particular Schwarzschild, Kerr, Reissner–Nordstrom, ...)

Fine tuning / Naturalness problem in GR

• true for electric and scalar tides too

• beyond linear regime

• for some ℓ in $D \neq 4$

× not true in AdS

× beyond pure GR (GREFT, modifications of GR)

× frequency corrections $\Rightarrow$ $\neq 0$ response

(*) Most common explanation why something is $=0$ is

symmetry $\left(\begin{array}{l} \text{e.g. } f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots a_{2n} x^{2n} \\ f(-x) = f(x) \Rightarrow a_1 = a_3 = \dots = a_{2n-1} = 0 \end{array} \right)$

Symmetries of (e.g.) Schwarzschild are
- spherical symm $SO(3)$
- time translations

but these do not explain $\lambda_\ell = 0$

$\Rightarrow$ some $\boxed{\text{"hidden" symmetries}}$?

– in fact YES!

Ladder symmetries (Hui et al 2021 2105.01069)

- example scalar tides of Schwarzschild

(see 2203.08832)

$$S = -\frac{1}{2}\int d^4x \sqrt{-g}\,(\nabla_\mu\phi)(\nabla^\mu\phi) = \Big/ \phi = \sum_{\omega,\ell m} e^{-i\omega t} Y_{\ell m}\,\phi_{\ell m}(r) \Big/ =$$

$$= \dots\dots = \frac{1}{2}\int dt\,dr\,d\Omega \left[\frac{r^4}{\Delta}(-\omega^2)\phi^2 - \Delta(\partial_r\phi)^2 - (\nabla_A\phi)(\nabla^A\phi) \right]$$

$\Delta \equiv r(r-r_s)$

- near-zone approx $r_s \leq r \ll \frac{1}{\omega}$

$$\frac{\omega^2 r^4}{\Delta} \approx \frac{\omega^2 r_s^4}{\Delta}$$

$$S \overset{\text{near zone}}{\approx} -\frac{1}{2}\int d^4x \sqrt{g_{eff}}\, g^{\alpha\beta}_{eff}(\partial_\alpha \phi)(\partial_\beta \phi)$$

$$g^{eff}_{\mu\nu} = \underbrace{-\frac{\Delta}{r_s^2}dt^2 + \frac{r_s^2}{\Delta}dr^2}_{} + \underbrace{r_s^2 d\Omega^2_{S_2}}_{}$$

this is metric of $\underline{AdS_2} \times \underline{S_2}$

for "small ω" scalar effectively sees $AdS_2 \times S_2$ background

but this is more symmetric than Schwarzschild!

- 6 Killing vectors (time transl + 3 rot. + 2 extra)
- 9 Conformal Killing vectors (CKV)
  - ~ $\nabla_\mu K_\nu + \nabla_\nu K_\mu = \frac{1}{2} g_{\mu\nu}(\nabla_\alpha K^\alpha)$
  - ~ they preserve angles but not lengths and areas
  - ~ CKV are symmetries because it turns out ϕ is conformally coupled

KV + CKVs $\rightarrow$ $SO(3,1)$ algebra!

- subset of these preserves near-zone approx
- works for Kerr too
- similar symms for EM and gravity tides

- other proposals in Literature („Love symm", KdV inspired...)

- $S = \int d^4x \sqrt{-g}\, R[g] \rightarrow \partial_t \rightarrow 0$ („static limit")

change of variables:

$$V \equiv g_{00}\,,\quad f_{ij} \equiv \partial_i\left(\frac{g_{0j}}{g_{00}}\right) - \partial_j\left(\frac{g_{0i}}{g_{00}}\right),\quad h_{ij} = g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}$$

EoMs $\Rightarrow f_{ij} = \sqrt{h}\, \varepsilon_{ijk} \frac{\nabla_k \phi}{V^{3/2}}$

h_{ij}, $\mathcal{E} \equiv V + i\phi$

- rescale V and ϕ
- translate ϕ
- Ehlers transf.

$$V \pm \phi \rightarrow \frac{V \pm \phi \pm b(V^2 - \phi^2)}{(1 \pm b\phi)^2 - b^2 V^2}$$

Ernst equation

$$\boxed{\mathrm{Re}(\mathcal{E})\, \nabla^2 \mathcal{E} = (\nabla \mathcal{E})^2}$$

$\Leftrightarrow SO(2,1)$ nonlinear sigma model

* static and axisymmetric
  $\Rightarrow$ even simpler $\rightarrow$ ∞ symmetries (Geroch)
  integrable system

[PRD „Connection between the nonlinear σ model and the Einstein eqs of GR" Norma Sanchez (1982)]

Nonlinearities

* **Self Force** $\quad \frac{m_2}{m_1} \ll 1 \sim 10^{-3} - 10^{-6}$ SMBHs (target for LISA)

solve full Einsteins eqs (semi-analytically)

in $\frac{m_2}{m_1}$ small expansion

1SF $\qquad$ 2SF

$$g_{\mu\nu} = \text{Ker} + \frac{m_2}{m_1}(\text{pert}) + \left(\frac{m_2}{m_1}\right)^2 [\ldots]$$

$$x^\mu(\tau) = \begin{array}{c}\text{Kerr}\\ \text{geodesic}\end{array} + \frac{m_2}{m_1}\delta x^\mu + \ldots$$

from perts.

[BHPT and gravitational Self-Force
Pound & Wardell]

* **Nonlinear QNMs** (2406.14611)

$$g_{\mu\nu} = g_{\mu\nu}^{Sch} + \epsilon h_{\mu\nu}^{(1)} + \epsilon^2 h_{\mu\nu}^{(2)}$$

$$\omega_{12}^{(2)} = \omega_1 \pm \omega_2$$

$$\hat{\mathcal{D}} h_{\mu\nu}^{(1)} = 0$$

$$\mathcal{D} h_{\mu\nu}^{(2)} = \mathcal{O}\left((h^{(1)})^2\right) \qquad R_{1,2,12} \quad \frac{A_{12}^{(2)}}{A_1^{(1)} A_2^{(1)}}$$

$1 \sim (n_1 l_1 m_1 p_1)$
$2 \sim (n_2 l_2 m_2 p_2)$ $\quad 12 \sim (n_1 l_1 m_1 p_1) \times (n_2 l_2 m_2 p_2)$

* **Nonlinear tides** $\longrightarrow$ Nonlinear Love numbers

$$h_{00} \sim \ldots + \varepsilon r^2 + \ldots + \varepsilon \frac{\lambda}{r^3} + \ldots + \varepsilon^2 r^4 + \ldots + \varepsilon^2 \frac{\rho}{r^3} + \ldots$$
$$+ \mathcal{O}(\varepsilon^3) + \ldots$$

(2410.03542)